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DIFFERENTIAL AND DIFFERENCE EQUATIONS

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ABSTRACTS

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CONTENTS

BIFURCATIONS IN IMPULSIVE SYSTEMS

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Many evolutionary processes in the real world are characterized by sudden changes at certain times. These changes are called to be impulsive phenomena [1, 3], which are widespread in modeling in mechanics, electronics, biology, neural networks, medicine, and social sciences. An impulsive differential equation is one of the basic instruments to understand the role of discontinuity better for the real world problems.

If the impulses occur at fixed times, the mathematical model of this process will be given by the following impulsive system [1]

$$\dot{x} = f(t, x, \alpha), \quad t \neq t_k, \quad x(t_k^+) = h_k(x(t_k), \alpha), \tag{1}$$

where $\{t_k\}_{k\in\mathbb{Z}}$ is a strictly increasing real sequence of impulse times that is unbounded on $\mathbb{R}$, $x(t^+) = \lim_{s\to t+0} x(s)$, α is a numeric or vector parameter, $f(t, 0, \alpha) = 0$, $h_k(0, \alpha) = 0$. Introduce a sequence of postive numbers $\{\theta_k = t_{k+1} - t_k\}_{k\in\mathbb{Z}}$. We take the typical convention that piecewise smooth solutions of the impulsive system are continuous from the left.

We will discuss a local bifurcations in the parameter-dependent system (1). If system (1) is periodic, i.e. $\theta_{k+p} = \theta_k$, $h_{k+p} = h_k$, $f(t + T, x) = f(t, x)$ and $t_{k+p} = t_k + T$ (or $\theta_1 + \dots + \theta_p = T$) then the problem can be reduced to a problem in discrete time (see, for instance, [2, 3]).

Here we study a case, when $\{\theta_k\}_{k\in\mathbb{Z}}$ is almost periodic sequence. Moreover, we admit for simplicity that the system (1) is "autonomous" system of order 2:

$$\dot{x} = A(\alpha)x + f(x, \alpha), \quad t \neq t_k, \quad x(t_k^+) = B(\alpha)x(t_k) + h(x(t_k), \alpha), \tag{2}$$

where $A \in \mathbb{R}^{2\times 2}$, $f(x, \alpha) = o(|x|)$, $h(x, \alpha) = o(|x|)$ as $|x| \to 0$. Solutions of the impulsive system are not continuous thus we need some definitions from [1].

A sequence $\{x_k\} \in \mathbb{R}^n$ is called almost periodic if for any $\varepsilon > 0$ there exists a relatively dense set of its ε-almost periods, i.e. there exists such a natural number N that, for an arbitrary $s \in \mathbb{Z}$, there is at least one number p in the segment $[s, s+N]$, for which $\|x_{k+p} - x_k\| < \varepsilon$ for all $k \in \mathbb{Z}$. The family of the sequences $\{t_k^j = t_{k+j} - t_k\}_{k\in\mathbb{Z}}$, $j \in \mathbb{Z}$, will be called equipotentially almost periodic if for an arbitrary $\varepsilon > 0$ there exists a relatively dense set of ε-almost periods, that are common to all the sequences $\{t_k^j\}_{k\in\mathbb{Z}}$.

Let a function $\varphi\colon \mathbb{R} \to \mathbb{R}^n$ be piecewise continuous with first kind discontinuities at the points of a fixed sequence $\{t_k\}$ and the family of sequences $\{t_k^j\}$, $k, j \in \mathbb{Z}$, is equipotentially almost periodic. We call a function φ almost periodic if: (a) for any $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that if the points t' and t'' belong to the same interval of continuity and $|t' - t''| < \delta$, then $\|\varphi(t') - \varphi(t'')\| < \varepsilon$; (b) for any $\varepsilon > 0$ there exists a relatively dense set Γ of ε-almost periods such that if $\tau \in \Gamma$, then $\|\varphi(t+\tau) - \varphi(t)\| < \varepsilon$ for all $t \in \mathbb{R}$ which satisfy the condition $|t - t_k| > \varepsilon$.

Let $\theta > 0$ be a mean value of the almost periodic sequence $\{\theta_k\}$ then $\theta_k = \theta + \delta_k$ and the mean of the sequence $\{\delta_k\}$ equals to zero.

Consider the linear impulsive system

$$\dot{x} = A_0 x, \quad t \neq t_k, \quad x(t_k^+) = B_0 x(t_k), \tag{3}$$

where $A_0 = A(0)$, $B_0 = B(0)$. Suppose that $A_0 B_0 = B_0 A_0$ and a matrix $M = e^{\theta A_0} B_0$ has eigenvalues $\rho_{1,2} = e^{\pm i\gamma}$, $0 < \gamma < \pi$.

All solutions of the system (3) are bounded. Taking into account known results [1] on existence of almost periodic solutions in impulsive systems (1) we study conditions when in a small neighbourhood of $x = 0$ the system (2) has a unique almost periodic solution which tends to zero when α tends to 0.

The theory of bifurcations in an impulsive system of arbitrary order is based on the center manifold method. After reducing the problem to equations on the center manifold, we can apply our approach.

It should be noted the first results on the study of bifurcations of solutions of non-periodic impulsive systems and functional differential impulsive systems presented in [3].

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CAUCHY PROBLEM FOR THE WAVE EQUATION WITH A NONSMOOTH RIGHT-HAND SIDE OF A SPECIAL FORM

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Let $T > 0$, $K = \{(t,x) \in (0,T) \times R^n : |x| < T - t\}$. We consider the following problem

$$u_{tt}(t,x) - \Delta u(t,x) = F(t,x) = f(v(t,x)), \quad (t,x) \in K, \tag{1}$$

$$u(0,x) = 0, \quad u_t(0,x) = 0, \quad |x| < T, \tag{2}$$

with bounded $f(z) \in C(R)$ and $v(t,x) \in W_2^2(K)$. We would like to find a regular solution $u(t,x) \in W_2^2(K)$ to this problem. Note, that in one-dimensional case it can be shown that the second derivatives $u_{tt}(t,x)$, $u_{xt}(t,x)$, $u_{xx}(t,x)$ are Lipschitz continuous provided $F(t,x)$ is also Lipschitz continuous (see, for example, [1]). In multidimesional case a regular solution exists if $F_t(t,x) \in L_2(K)$ or $\nabla F(t,x) \in L_2^n(K)$. It turns out that a regular solution exists even for $f(z) \in C(R)$ if some monotonicity condition for the function $v(t,x)$ is fulfilled.

Let's denote $\Omega_t = \{x : |x| < T - t\}$.

Theorem. *Suppose that* $|f(z)| \leqslant M$, $v(t,x) \in W_2^2(K)$ *and for some* $0 < \gamma < 1$ *and* $\delta > 0$

$$\gamma v_t(t,x) > |\nabla v(t,x)| + \delta, \quad \textit{for a.e. } (x,t) \in K.$$

Then there exists a unique solution to the problem (1)–(2) *and for a.e.* $t \in (0,T)$

$$\|u_{tt}\|_{L_2(\Omega_t)} + \|u_t\|_{W_2^1(\Omega_t)} + \|u\|_{W_2^2(\Omega_t)} \leqslant C(T,\gamma,\delta) M \|v\|_{W_2^2(K)}.$$

The main idea of the proof is to use some version of Duhamel's principle.

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ON ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO HIGHER-ORDER QUASILINEAR DIFFERENTIAL EQUATIONS FOR DIFFERENT TYPES OF PERTURBATIONS

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We study the problem of asymptotic behavior of solutions to equations

$$y^{(n)}(x) + \sum_{j=0}^{n-1} a_j(x) y^{(j)}(x) + p(x)\,|y(x)|^k \operatorname{sgn} y(x) = f(x)$$

with $n \geq 2,\ k > 1$, and continuous functions p, f and a_j. We will consider this equation as a perturbation of more simple equation with $f = 0$. This equation, in its turn, we will consider as a perturbation of the equation with $p = 0$ or/and $a_j = 0$.

Some previous results are formulated in [1–2]. In particular, the asymptotic behavior of its solutions vanishing at infinity is described.

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NON-UNIQUENESS OF CYCLES IN 3D MOLECULAR REPRESSILATOR MODELS

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The following 3D dynamical system describes functioning of a simplest molecular repressilator, see [1–3] for details and interpretations:

$$\dot{x_1} = L_1(x_3) - k_1x_1; \quad \dot{x_2} = L_2(x_1) - k_2x_2; \quad \dot{x_3} = L_3(x_2) - k_3x_3. \quad (1)$$

1. In the case of piecewise-linear monotonically decreasing non-negative functions L_j of the type

$$L_j(w) \equiv b_j - p_j\ell_{j-1} \quad \text{for} \quad 0 \le w \le \ell_{j-1};$$
$$L_j(w) = b_j - p_jw \quad \text{for} \quad \ell_{j-1} \le w \le b_jp_j^{-1}; \quad (2)$$
$$L_j = 0 \quad \text{for} \quad w \ge b_jp_j^{-1}$$

we find conditions of existence of several cycles in phase portraits of the systems of the types (1), (2). Here and below, all parameters and variables are positive, $j = 1, 2, 3$, and $j - 1 := 3$ for $j = 1$.

2. Let $\mathcal{L} = \mathcal{L}(w)$ be a three-step function defined by

$$\mathcal{L}(w) = 2a \quad \text{for} \quad 0 \le w < a - \varepsilon; \quad \mathcal{L}(w) = a + \varepsilon \quad \text{for} \quad a - \varepsilon \le w < a;$$

$$\mathcal{L}(w) = a - \varepsilon \quad \text{for} \quad a \le w < a + \varepsilon; \quad \mathcal{L}(w) = 0 \quad \text{for} \quad w \ge a + \varepsilon. \quad (3)$$

The cube $Q := [a - \varepsilon, a + \varepsilon] \times [a - \varepsilon, a + \varepsilon] \times [a - \varepsilon, a + \varepsilon]$ is invariant with respect to positive shifts along trajectories of the system (1), (3). In the symmetric case $k_1 = k_2 = k_3 = 1$, $L_1 = L_2 = L_3 = \mathcal{L}$, we obtain

Theorem. *If $a > 1$ and ε is sufficiently small then the system*

$$\dot{x_j} = \mathcal{L}(x_{j-1}) - x_j; \qquad j = 1, 2, 3;$$

has at least two different cycles.

Exactly one of these cycles $\mathcal{C}_1$ is contained in the invariant "small" cube Q; this cycle is stable. As in [4, 5], one can construct in Q an invariant surface Σ such that $\mathcal{C}_1 \subset \Sigma \subset Q$, and an invariant foliation such that the invariant surface Σ is one of its leaves.

Other cycles of the system (1), (3) are not local, they do not intersect Q.

REMARK. Previously, non-uniqueness of cycles was detected in similar models of molecular repressilators in higher-dimension cases only, such as 5D, 15D, 18D etc., see [6, 7].

The authors were supported by the Russian Science Foundation (project no. 23-21-00019).

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ON PROPERTIES OF THE NOETHERICITY OF POLYHARMONIC OPERATORS IN WEIGHTED SOBOLEV SPACES

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We consider equation

$$\Delta^m u = f(x), \quad x \in R^n, \tag{1}$$

where $f(x) \in L_p(R^n)$, *supp* f is compact.

The work is concerned with the question about existence and uniqueness of solutions to equation (1) in the class of weighted Sobolev spaces $W^{2m}_{p,\sigma}(R^n)$ suggested by G. V. Demidenko in his work [1].

Definition. A function $u(x)$ belongs to weighted Sobolev space $W^{2m}_{p,\sigma}(R^n)$ if there exist generalized derivatives

$$D^\beta_x u(x), \quad |\beta| \le 2m,$$

of $u(x)$ in R^n, and

$$\left\|(1+|x|^{2m})^{-\sigma(1-|\beta|/2m)} D^\beta_x u(x),\, L_p(R^n)\right\| < \infty.$$

The norm in the space $W^{2m}_{p,\sigma}(R^n)$ is defined as

$$\left\|u(x),\, W^{2m}_{p,\sigma}(R^n)\right\| = \sum_{0\le|\beta|\le 2m} \left\|(1+|x|^{2m})^{-\sigma(1-|\beta|/2m)} D^\beta_x u(x),\, L_p(R^n)\right\|.$$

The research method is based on the construction of approximate solutions to equation (1) using the method of integral representation of summable functions $f(x) \in L_p(R^n)$ suggested by S. V. Uspenskii in his work [2].

Some important results, extending those of G. V. Demidenko in [1], have been achieved, among them:

Theorem 1. *If* $n > 2m$, $\sigma \ge \frac{n}{2mp}$, *then* $\forall\, f(x) \in L_p(R^n)$, *supp* f *is compact,* $\exists\, u(x) \in W^{2m}_{p,\sigma}(R^n)$ *— a solution to equation* (1), *and the estimate takes place:*

$$\left\|u(x),\, W^{2m}_{p,\sigma}(R^n)\right\| \le c\, \|f(x), L_p(R^n)\|,$$

where $c = c\,(supp\, f)$*; for* $\sigma = \frac{n}{2mp}$ *the solution is unique.*

Theorem 2. *If* $n > 2m$, $\sigma > \frac{n}{2mp}$, *then the solution is defined up to polynomials* $P_j(x)$ *of degree not greater than* j*:*
if $\sigma \in \left(\frac{n}{2mp} + \frac{j}{2m}, \frac{n}{2mp} + \frac{j+1}{2m}\right]$, $j \geq 0$, *then*

$$ker\ \Delta^m = \{P_j(x)\},$$

i.e.

$$u(x) = u_{part.}(x) + P_j(x).$$

Theorem 3. *If* $n > 2m$, $\sigma \in [0, 1 - \frac{n}{2mp'}]$, N *is natural number, such that* $1 - \frac{n}{2mp'} - \frac{N}{2m} < \sigma \leq 1 - \frac{n}{2mp'} - \frac{N-1}{2m}$, *and*

$$\int_{R^n} x^\beta f(x)\, dx = 0, \quad 0 \leq |\beta| \leq N - 1,$$

then $\forall\, f(x) \in L_p(R^n)$, $supp\, f$ *is compact,* $\exists\,!\, u(x) \in W^{2m}_{p,\,\sigma}(R^n)$ *— the solution to equation* (1)*, and the estimate takes place:*

$$\left\|u(x),\, W^{2m}_{p,\,\sigma}(R^n)\right\| \leq c\,\|f(x),\, L_p(R^n)\|\,,$$

where $c = c\,(supp\, f)$.

It's worth mentioning that some results on solvability of elliptic equations in similar class of weighted Sobolev spaces have been achieved in the works of R. C. McOwen (see, for example, [3]).

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FINDING THE DISCONTINUITY SURFACES OF COEFFICIENTS OF THE TRANSPORT EQUATION

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Consider a non-stationary linear differential equation

$$\frac{\partial f(t,r,\omega,E)}{\partial t} + \omega\cdot\nabla_r f(t,r,\omega,E) + \mu(t,r,E)f(t,r,\omega,E) = J(t,r,\omega,E).$$

This equation specifically describes the process of particle transfer through a medium, with the following parameters: time is represented by the temporary variable t, which falls within the interval $t\in[0,T]$; spatial location is denoted by the variable r, within a convex bounded area G; the unit vector ω belongs to the set $\Omega=\{\omega\in\mathbb{R}^3 : |\omega|=1\}$; energy is represented by E, falling within the interval $I=[E_1,E_2]$, with $E_1>0$ and $E_2<\infty$. The function $f(t,r,\omega,E)$ signifies the particle flux at time t, position r, with energy E, moving towards ω. The characteristics of the environment G are defined by the functions μ and J. In addition to the equation, we have initial and boundary conditions that determine the density of the incoming flow h and the average outflow density H, with only the function H being known.

The primary objective is to identify the internal structure of the environment, denoted as G, by solving the problem of detecting discontinuity surfaces in the coefficients μ and J within the equation. This research builds upon the work of D. S. Anikonov [1]. The initial step involves studying the direct problem of computing the flux density, represented as f, based on the initial conditions and the incident flow density h. This problem bears similarities to A. I. Prilepko's work with continuous coefficients [2].

Furthermore, we introduce a special function, denoted as $Ind(r)$, defined as:

$$Ind(r) = \left| \nabla \int_d^T \int_\Omega H(t, r+d(r,\omega)\omega,\omega)\,d\omega dt \right|.$$

This function depends on known data, where $d(r,\omega)$ represents the distance from point r to the boundary ∂G in the direction of ω, and d signifies an area diameter of G. It has been proven that the function Ind takes unbounded values only on the desired surfaces.

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ON POSITIVENESS OF THE CAUCHY FUNCTION AND THE FUNDAMENTAL SOLUTION FOR A NEUTRAL DIFFERENTIAL EQUATION

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We study linear autonomous differential equation of neutral type

$$\dot{x}(t) - a\dot{x}(t-h) + bx(t) + cx(t-h) = f(t), \quad t \geq 0, \tag{1}$$

where $a, b, c > 0$, $h > 0$, f is a locally integrable function. This equation arises in applications. For example, it describes dynamics of cell population, motion of 2-dimensional elastic plates with friction, and ultrasonic flaw detection. This equation is also interesting from the theoretical point of view, which is confirmed by a large number of purely theoretical studies.

Transform the time scale $t \mapsto ht$ and the coefficients $b \mapsto hb$, $c \mapsto hc$, and rewrite equation (1) in an equivalent form, which is more convenient for our further study:

$$(I - aS)\dot{x}(t) + (bI + cS)x(t) = f(t), \quad t \in \mathbb{R}_+, \tag{2}$$

where

$$(Sy)(t) = \begin{cases} y(t-1), & \text{if } t \geq 1, \\ 0, & \text{if } t < 1. \end{cases}$$

As is known [1], equation (2) with an initial condition admits a unique solution in the class of locally absolutely continuous functions; moreover, this solution has the following form [1, 2]:

$$x(t) = X(t)x(0) + \int_0^t Y(t-s)f(s)\,ds, \quad t \in \mathbb{R}_+. \tag{3}$$

The function X is called the *fundamental solution* and the function Y is called the *Cauchy function.* Formula (3) is usually called the *Cauchy formula.*

The fundamental solution X is a locally absolutely continuous function and is uniquely dened as the solution to equation (2) for $f = 0$ satisfying the initial condition $x(0) = 1$.

On each interval $(n, n+1)$, $n \in \mathbb{N}_0$, the function Y is absolutely continuous but each point $t = n$ is a discontinuity of the rst kind; moreover, we have $Y(n+0) - Y(n-0) = a^n$ [2]. As is shown in [2, 3], the function Y is uniquely determined in terms of the function X by the equality

$$(I - aS)Y(t) = X(t), \quad t \in \mathbb{R}_+.$$

It follows from (3) that the behavior of any solution of equation (2) is completely determined by properties of X and Y. In [4] we find necessary and sufficient conditions for exponential stability and represent them in geometric terms as a domain in the space of parameters. In this paper we study the positiveness of the functions X and Y.

The characteristic function of (2) is

$$g(p) = p(1 - ae^{-p}) + b + ce^{-p}, \quad p \in \mathbb{C}.$$

Lemma. *The operator $I - aS$ is positively invertible on each finite segment.*

Theorem 1. *The fundamental solution X of equation* (2) *is positive if and only if the Cauchy function Y of equation* (2) *is positive.*

Theorem 2. *The fundamental solution X of equation* (2) *is positive if and only if the function g has at least one real zero.*

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A NUMERICAL METHOD FOR CONSENSUS CONTROL OF LINEAR DELAY MULTI-AGENT SYSTEMS

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In this paper, the consensus control problem of leader-following linear multi-agent systems with the input delay is investigated. We present a necessary and sufficient condition for the leader-following consensus of linear multi-agent systems with the input delay. Based on this necessary and sufficient condition, the leader-following consensus problem can be transformed into a distributed non-convex optimization problem.

Consider a multi-agent system consisting of N follower agents and a leader. The dynamics of each follower agent is

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \tag{1}$$

where $x_i(t) \in \mathbb{R}^d$ is the state of the agent i, $u_i(t) \in \mathbb{R}^m$ is the control input of the agent i, and $A \in \mathbb{R}^{d\times d}$ and $B \in \mathbb{R}^{d\times m}$ are known constant matrices. The leader, labeled as $i = 0$, has linear dynamics as

$$\dot{x}_0(t) = Ax_0(t), \tag{2}$$

where $x_0(t) \in \mathbb{R}^d$ is the state of the leader. Obviously, the leader's dynamics is independent of others, which acts as an external input to steer follower agents.

The control law for the agent i is given as

$$u_i(t) = K \sum_{j\in\mathcal{N}_i} a_{ij}[x_i(t-\tau) - x_j(t-\tau)]$$

$$+Kg_i[x_i(t-\tau) - x_0(t-\tau)], \quad i = 1, \ldots, N, \tag{3}$$

where $\tau > 0$ is the input delay in the network and $K \in \mathbb{R}^{m\times d}$ is a feedback gain matrix to be designed later.

In order to analyze the leader-following consensus of the system (1)–(2), the error variable between the state x_i and x_0 is denoted by $\delta_i = x_i - x_0$, we obtain

$$\dot{\delta}_i(t) = A\delta_i(t) + BK \sum_{j\in N_i} a_{ij} \left[\delta_i\left(t-\tau\right) - \delta_j\left(t-\tau\right)\right] + BKg_i\delta_i\left(t-\tau\right), \quad (4)$$

where $i = 1, \ldots, N$.

Definition. The transition matrix of $\delta_i(t)$ in system (4) is defined as $F_i(K,t)$, which is the solution of the matrix differential equation

$$\dot{F}_i(K,t) = AF_i(K,t) + BK \sum_{j\in N_i} a_{ij} \left[F_i\left(K, t-\tau\right) - F_j\left(K, t-\tau\right)\right]$$
$$+BKg_iF_i\left(K, t-\tau\right)$$

under the condition

$$\begin{cases} F_i(K,0) = \begin{pmatrix}\mathbf{0}_{d\times(i-1)d} & \mathbf{I}_{d\times d} & \mathbf{0}_{d\times(N-i)d}\end{pmatrix}_{d\times Nd}, \\ F_i(K,t) = \mathbf{0}_{d\times Nd} \quad \text{for} \quad t<0, \end{cases}$$

where $i = 1, \ldots, N$.

Theorem. *Under the control law* (3), *the considered multi-agent system* (1)–(2) *achieves the leader-following consensus if and only if there exists a feedback gain matrices K such that*

$$\lim_{T\to\infty} \sum_{i=1}^{N} \int_0^T \lambda \left\|F_i(K,\sigma)\right\|_F^2 d\sigma + \left\|K\right\|_F^2$$

exists and is finite, where λ is a positive constant.

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BOUNDARY VALUE PROBLEMS FOR DEGENERATE INTEGRO-DIFFERENTIAL EQUATIONS

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The paper studies the solvability of boundary value problems for the second-order integro-differential equations with integral terms of Volterra type

$$u_{tt}(x,t) - h(t)\Delta u(x,t) + c(x,t)u(x,t) = \int\limits_0^t R(t,\tau)(Bu)(x,\tau)d\tau + f(x,t).$$

A peculiarity of the equations under study is that the main part of the equations is a degenerate hyperbolic operator, and an operator can degenerate in both a characteristic and non-characteristic way.

For the problems under study, theorems of existence and uniqueness of regular solutions, i.e., solutions having all Sobolev generalized derivatives included in the equations, are proved.

THE ESTIMATES OF THE ALEXANDROV'S n-WIDTH OF A COMPACT SET OF C^∞-SMOOTH FUNCTIONS ON A FINITE SEGMENT

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Two-sided qualified estimates of the Alexandrov's n-width of a compact set of infinitely smooth functions are obtained, boundedly embedded in the space of continuous functions on a finite segment.

This work was carried out within the framework of the state contract of the Sobolev Institute of Mathematics (project no. FWNF-2022-0008).

INFLUENCE OF THE PSEUDOSPECTRUM OF COEFFICIENT MATRIX OF A LINEAR ODE SYSTEM ON THE LOCAL GROWTH OF SOLUTIONS

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Studying the spectrum of a linearized differential operator is one of the most common methods for studying stability. However, some data from physical experiments indicate that the spectrum of the linearized operator does not accurately predict the development of instability. Some publications have previously suggested that this may be due to the size and position of the spots in the so-called pseudospectrum [1].

The pseudospectrum (or ε-spectrum) is a set of complex numbers λ that satisfy the estimate $\|(A - \lambda I)^{-1}\| \geq 1/\varepsilon$ at a fixed $\varepsilon > 0$. We assume that eigenvalues united by one spot of the pseudospectrum may have properties similar to those of multiple eigenvalues. Based on this assumption, we created two algorithms for constructing the initial conditions of the Cauchy problem for a system of linear ODEs, under which the solution will have a local maximum in time. One of the algorithms is based on the use of eigenvectors. Another involves using a matrix spectrum dichotomy [2] to separate eigenvalues belonging to the same spot of the pseudospectrum.

Using these algorithms, solutions to the system of Navier–Stokes equations, linearized in the vicinity of the Poiseuille flow, are constructed, which increase approximately 50 times in the initial time interval. In a similar way, solutions of one flutter model that grow locally by 80 times were constructed.

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A MODIFIED QUADRATIC INTERPOLATION METHOD FOR ROOT FINDING

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A modification of the quadratic interpolation method for finding the root of a continuous function is proposed. Two quadratic interpolation polynomials are simultaneously constructed. It is shown that if the third derivative of the original function does not change sign on the considered interval of localization of the required root, then the root lies between the roots of the quadratic functions. This allows to significantly narrow the localization interval and reduce the number of steps to calculate the root with a given accuracy. The proposed modification of the quadratic interpolation method is used in the problem of calculating isolines when modeling the hill diagram of hydraulic turbines.

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THE CAUCHY PROBLEM FOR ONE PSEUDOHYPERBOLIC SYSTEM

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In the paper we consider the Cauchy problem for one system unsolvable with respect to the highest time derivative

$$\begin{pmatrix} I-\alpha D_x^2 & 0 & a_1 \\ 0 & I-\alpha D_x^2 & -a_2 \\ ca_1 & -ca_2 & I-\alpha D_x^2 \end{pmatrix} D_t^2 \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}$$

$$+\beta \begin{pmatrix} D_x^4 & 0 & 0 \\ 0 & D_x^4 & 0 \\ 0 & 0 & D_x^4 \end{pmatrix} \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} = F(t,x), \tag{1}$$

$$(t,x) \in R_+^2 = \{t>0,\, x \in R\},$$

$$\begin{pmatrix} u \\ v \\ \theta \end{pmatrix}\Bigg|_{t=0} = \Phi(x), \quad D_t \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}\Bigg|_{t=0} = \Psi(x),$$

where $\alpha,\, \beta > 0$, $0 < c(a_1^2 + a_2^2) < 1$. This system describes transverse flexural-torsional vibrations of an elastic rod [1].

A classification of equations unsolvable with respect to the highest order derivative was introduced in [2]. In particular, a class of pseudohyperbolic equations was introduced and the Cauchy problem for them was studied. Further studies of the solvability of the Cauchy problem for pseudohyperbolic equations were carried out in [3–5].

The system under consideration belongs to the class of pseudohyperbolic systems. There is no general theory of solvability of the Cauchy problem for this class of systems . There are only particular results for certain systems. The following theorem is proven.

Theorem. *Let* $e^{-\gamma t}F(t,x) \in W_2^{0,1}(R_+^2)$, $\gamma > 0$, $\Phi(x) \in W_2^4(R)$, $\Psi(x) \in W_2^3(R)$. *Then the Cauchy problem* (1) *has a unique solution*

$$U(t,x) = (u(t,x), v(t,x), \theta(t,x))^T$$

such that $e^{-\gamma t}U(t,x) \in W_2^{2,4}(R_+^2)$, $e^{-\gamma t}D_t^2 D_x^2 U \in L_2(R_+^2)$, *and the following inequality holds:*

$$\|e^{-\gamma t}U(t,x), W_2^{2,4}(R_+^2)\| + \|e^{-\gamma t}D_t^2 D_x^2 U(t,x), L_2(R_+^2)\|$$

$$\leq c(\gamma)\Big(\|\Phi(x), W_2^4(R)\| + \|\Psi(x), W_2^3(R)\| + \|e^{-\gamma t}F(t,x), W_2^{0,1}(R_+^2)\|\Big),$$

where $c(\gamma)$ *is a constant depending on the system coefficients and* γ.

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ON THE SOLVABILITY OF THE CAUCHY PROBLEM FOR SECOND ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MIXED RESTRICTIONS ON FUNCTIONAL NON-VOLTERRA OPERATORS

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We consider linear second order functional differential equations without the argument delay condition. Such non-Volterra or delayed and advanced equations are interesting from a theoretical point of view, but more and more mathematical models using such equations are appearing. The initial problem (or Cauchy problem) for such equations is generally not uniquely solvable under natural conditions. Therefore, the conditions for the unique solvability of the Cauchy problem are important and interesting. In the simplest case of an equation with concentrated deviation, our object looks like this:

$$\ddot{x}(t) = q(t)x(h(t)) + f(t), \quad t \in [0, 1], \tag{1}$$

where q, $f : [0, 1] \to \mathbb{R}$ are integrable functions, $h : [0, 1] \to \mathbb{R}$ is a measurable deviation of the argument.

Let us consider the most general case of functional differential equations, which can be conveniently written in the operator form:

$$\ddot{x}(t) = (T^+x)(t) - (T^-x)(t) + f(t), \quad t \in [0, 1].$$

Here T^+ and T^- are linear positive operators acting from the space of real continuous functions into the space of real integrable functions (positive operators map non-negative functions into non-negative ones).

We find conditions for the unique solvability of the Cauchy problem

$$\begin{cases} \ddot{x}(t) = (T^+x)(t) - (T^-x)(t) + f(t), & t \in [0, 1], \\ x(0) = c_0, \quad \dot{x}(0) = c_1, \end{cases} \tag{2}$$

for all positive operators T^+ and T^- satisfying the equalities

$$(T^+\mathbf{1})(t) = p^+(t), \quad (T^-\mathbf{1})(t) = p^-(t), \quad t \in [0, 1],$$

where p^+ and p^- are two given non-negative integrable functions, $\mathbf{1}$ is the unit function, $\mathbf{1}(t) = 1$ for all $t \in [0,1]$, c_0, $c_1 \in \mathbb{R}$, f is integrable. By imposing various restrictions on the functions p^+ and p^-, we can obtain various conditions for the solvability of problem (2).

All known solvability conditions of this kind for many boundary value problems were obtained under the same types of restrictions on the operators T^+, T^-, that is only under pointwise restrictions or only under integral ones. We can obtain solvability conditions under mixed restrictions, when pointwise restrictions are imposed on the action of one of the operators T^+, T^-, and integral restrictions are imposed on the other operator.

Let us present several obtained statements.

Theorem 1. *Let constants $\mathcal{P}^+ \geq 0$, $\mathcal{P}^- \geq 0$ be given. Cauchy problem* (2) *is uniquely solvable for all linear positive operators T^+, T^- such that*

$$(T^-\mathbf{1})(t) \leq \mathcal{P}^-,\ t \in [0,1], \quad \int_0^1 (1-s)(T^+\mathbf{1})(s)\,ds \leq \mathcal{P}^+,$$

if and only if

$$\mathcal{P}^+ < 1, \quad \mathcal{P}^- < 8(1+\sqrt{1-\mathcal{P}^+}).$$

Theorem 2. *Let $\alpha \geq -1$. Let constants $\mathcal{P}^+ \geq 0$, $\mathcal{P}^- \geq 0$ be given. Cauchy problem* (2) *is uniquely solvable for all linear positive operators T^+, T^- such that*

$$\int_0^1 (1+\alpha s)(T^-\mathbf{1})(s)\,ds \leq \mathcal{P}^-, \quad \int_0^1 (1-s)(T^+\mathbf{1})(s)\,ds \leq \mathcal{P}^+,$$

if and only if

$$\mathcal{P}^+ < 1, \quad \mathcal{P}^- - \mathcal{P}^+ + 1 \leq \left(1+\sqrt{1+\alpha}+\sqrt{1-\mathcal{P}^+}\right)^2.$$

Note that the sign of the coefficient q in equation (1) is not required to be preserved, and no conditions are imposed on the deviation of the argument. The resulting solvability criteria are unimprovable in the sense that if they are not satisfied, then there are equations with allowed parameters for which the Cauchy problem is not uniquely solvable. We find solvability conditions that cannot be obtained by the contraction mapping method.

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NONLOCAL PROBLEMS WITH INTEGRAL CONDITIONS FOR LINEAR DIFFERENTIAL EQUATIONS OF THE THIRD ORDER

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We study the solvability of nonlocal problems, i.e., problems with integral conditions for linear ordinary differential equations of the third order

$$y''' + c(t)y = f(t).$$

Previously, when studying the solvability of such nonlocal problems, it was used either an approach based on the transition to problems with "semi-integral" conditions (by applying the corresponding integral operator to the equation), or an approach based on the representation of solutions using basic functions of equation $y''' + c(t)y = 0$. Our work assumes a new approach, which differs from the approaches of predecessors. We give new conditions for the solvability of the studied nonlocal problems.

STABILITY OF TRANSONIC SHOCKS PAST 3-D WEDGES

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Stability of transonic shocks is a difficult subject with a long history dating back to Prandtl, who conjectured in 1936 that out of two types of shocksstrong and weak shocks, the weak ones are stable. I will talk about the stability of three dimensional transonic shocks governed by the 3-D potential flow equation. It is showed that for a piecewise constant weak transonic flow, if the incoming flow and the wedge are slightly perturbed, there exist a unique weak transonic shock and downstream subsonic solution, which is also a small perturbation from the background solution. The connection between the shock condition and the elliptic estimates will be explained. I will also introduce a recent result about the stability of strong transonic shocks over 3-D wedges. We show that the strong shocks near normal shock regime are stable.

EFFECTIVE TESTS FOR ASYMPTOTIC PROPERTIES OF SOLUTIONS TO FIRST-ORDER DELAY DIFFERENTIAL EQUATIONS

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A. D. Myshkis was the first to systematically investigate conditions for solutions to differential equations with aftereffect to have certain asymptotic properties. In particular, it was he who first proved a number of statements related to the oscillation constant $1/e$ and found the stability constant $3/2$ [1]. Extensions and sharpenings of Myshkis's results constitute the important directions in the theory of delay differential equations, which are still being actively developed. We consider some achievements in these directions.

The most fundamental results by Myshkis were first obtained for the linear nonautonomous equation of the first order

$$\dot{x}(t) + a(t)x(h(t)) = 0, \quad t \geq 0, \tag{1}$$

where $h(t) \leq t$, and then extended to some more general equations. We consider results related to the case $a(t) \geq 0$, which is called by Myshkis *an equation of stable type*. The most perfect generalizations of these results for equation (1) are the following.

Suppose that $a(t) \geq 0$ for all $t \geq 0$, and $h(t) \to +\infty$ as $t \to +\infty$.

Theorem 1 [2]. *If* $\varliminf\limits_{t\to+\infty} \int_{h(t)}^{t} a(s)\,ds > 1/e$, *then all solutions to equation* (1) *oscillate. If* $\varlimsup\limits_{t\to+\infty} \int_{h(t)}^{t} a(s)\,ds \leq 1/e$, *then equation* (1) *has a nonoscillatory solution.*

Theorem 2 [3, 4]. *If* $\sup\limits_{t\geq T} \int_{h(t)}^{t} a(s)\,ds \leq 3/2$ *for some* $T > 0$, *then all solutions to equation* (1) *are uniformly stable. If* $\int_0^\infty a(s)\,ds = \infty$ *and* $\varlimsup\limits_{t\to+\infty} \int_{h(t)}^{t} a(s)\,ds < 3/2$, *then all solutions to equation* (1) *are asymptotically stable.*

We consider generalizations of Theorems 1 and 2 for equations of more general forms than (1). The most interesting cases for us are the equation with several delays

$$\dot{x}(t) + \sum_{k=1}^{n} a_k(t) x(h_k(t)) = 0, \quad t \geq 0, \tag{2}$$

and the first-order linear equation of the general form

$$\dot{x}(t) + \int_{h(t)}^{t} x(s)\, d_s r(t,s) = 0, \quad t \geq 0. \tag{3}$$

We consider equations of stable type which means that $a_k(t) \geq 0$ in (2), $r(t,\cdot)$ is increasing in (3), and $h(t) \to +\infty$ as $t \to +\infty$ in both (2) and (3). The first question on asymptotic properties of solutions to equation (2) that we consider is to find effective oscillation and stability tests (which are conditions expressed in terms of parameters of the equation under consideration) that equally take into account all terms of the sum in (2). The next question is generalization of such tests to the case of equations with distributed delay and, in the general case, to equation (3).

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DISCRETE ANALYTIC FUNCTIONS AND TAYLOR SERIES

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History. The notation of discrete analytic function on the Gaussian lattice $\mathbb{G} = \mathbb{Z}+i\,\mathbb{Z}$ was given by R. F. Isaacs [1]. He classified these functions into functions of first and second kind and investigated those of first kind. Further J. Ferrand [2] and R. J. Duffin [3] created the theory of discrete analytic functions of second kind (from now on: discrete analytic functions). Important results which are connected with a behaviour of discrete analytic and harmonic functions at infinity was obtained by S. L. Sobolev [4]. New combinatorial and analytical ideas to the theory were input by D. Zeilberger [5]. They were generalized by A. D. Mednykh [6]. An advance of nonlinear theory of discrete analytic functions based on usage of circle patterns began by W. Thurston [7] and his students [8, 9]. In that way an approximation with rapid convergence was obtained in the theory of conformal maps of Riemann surfaces.

Definition. From now on, $\mathbb{G} = \{x + i\,y : x, y \in Z\}$ is the Gaussian integer lattice and $\mathbb{G}^+ = \{x + i\,y \in \mathbb{G} : x \geq 0, y \geq 0\}$ is a part of Gaussian plane contained in the first quadrant. A complex function f defined on some subset $E \subset \mathbb{G}$ is called discrete analytic on E if for any square $\{z, z\ +\ 1, z\ +\ 1\ +\ i, z\ +\ i\}\ \subset\ E$ there holds:

$$\frac{f(z+1+i)-f(z)}{i+1} = \frac{f(z+i)-f(z+1)}{i-1}$$

or equivalently

$$\bar{\partial} f(z) = f(z) + if(z+1) + i^2 f(z+1+i) + i^3 f(z+i) = 0.$$

A discrete analytic function on all $\mathbb{G}^+$ is called entire discrete. Let us denote the set of all discrete analytic functions on E and on $\mathbb{G}^+$ by $\mathcal{D}(\mathcal{E})$ and $\mathcal{D}(\mathbb{G}^+)$ correspondingly.

Theorem 1. *Every discrete analytic function $f \in \mathcal{D}(\mathbb{G}^+)$ has a Taylor expansion in terms of $\pi_k(z)$:*

$$f(z) = \sum_{0}^{\infty} a_k \pi_k(z), \quad z \in \mathbb{G}^+.$$

Theorem 2. *Above mentioned expansion is not unique. More precisely,*

$$f(z)=\sum_{0}^{\infty}a_k\pi_k(z)\equiv 0,\quad z\in\mathbb{G}^+\quad\Leftrightarrow\quad F(s)=0,\quad s\in\mathbb{Z}.$$

Theorem 3. *A homomorphism* $\Theta:\mathcal{A}(U_R)\to\mathcal{D}(Q_R)$ *is "onto" and* $\Theta(F)\equiv 0\Leftrightarrow F(s)=0$, $s\in Z$, $|s|<R$. *In this case*

$$Ker\,\Theta=\langle F_N(\xi)\rangle=F_N\cdot(\mathcal{U}_{\mathcal{R}})$$

is a principal ideal in $\mathcal{A}(\mathcal{U}_{\mathcal{R}})$ *generated by function* $F_N(\xi)=\xi\prod_{k=1}^{N}(\xi^2-k^2)$, *where* $N=[R]$, *if* R *is non-integer and* $R-1$ *otherwise.*

Theorem 4. *Let* $f\in\mathcal{D}(\mathbb{G}^+)$. *Then there exists a function* $F(\xi)=\sum_{|k|=0}^{\infty}a_k\frac{\xi^k}{(1+i)^{|k|}}\in\mathcal{A}(\mathbb{C}^n)$ *such that* $f(z)=\sum_{|k|=0}^{\infty}a_k\pi_k(z)$ *and this expansion converge absolutely for all* $z\in\mathbb{G}^+$. *In addition,* $\Theta F=0\Leftrightarrow F(s)=0$ *for all* $s\in\mathbb{Z}^n$.

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ONE OF SYSTEMS OF DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS IN LINEAR PARTS

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We consider the following systems of nonlinear differential equations

$$\frac{dy}{dt} = A(t)y + f(t,y), \qquad -\infty < t < \infty, \tag{1}$$

where the entries of the matrix $A(t)$ of dimension $m \times m$ are continuous T-periodic functions, the continuous vector-function $f(t,y)$ satisfies the local Lipschitz condition in y and the conditions

$$f(t+T,y) \equiv f(t,y), \qquad \|f(t,y)\| \leq q(1+\|y\|)^{\omega}$$

hold, where $q > 0$ and $\omega \geq 0$ are constants.

We assume that the linear system

$$\frac{dy}{dt} = A(t)y, \qquad -\infty < t < \infty,$$

is *exponentially dichotomous*.

Our aim is to study conditions of existence of T-periodic solutions to (1) and stability of the solutions under small perturbations of coefficients and nonlinear terms.

The work is continue the research [1, 2].

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ON SOLVABILITY OF A BOUNDARY VALUE PROBLEM FOR PSEUDOHYPERBOLIC EQUATION WITH VARIABLE COEFFICIENTS

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We study correctness of a boundary value problem in a cylindrical domain $Q_T = \{(t,x) \in R^{n+1} : \ t \in (0,T), \ x \in G \subset R^n\}$ for the equation of the fourth order with variable coefficients

$$u_{tt} - \sum_{i=1}^{n} D_{x_i} \left(\alpha_i(x) D_{x_i} u_{tt}\right)$$

$$+ \sum_{i,j=1}^{n} D^2_{x_i x_j} \left(a_{ij}(x) D^2_{x_i x_j} u\right) - \sum_{i=1}^{n} D_{x_i}(b_i(x) D_{x_i} u) = f(t,x), \qquad (1)$$

where the coefficients of the equation are real-valued and sufficiently smooth functions, moreover

$$\alpha_i(x) \geq \alpha_i > 0, \ \ a_{ij}(x) \equiv a_{ji}(x) \geq a_{ij} > 0, \ \ b_i(x) \geq 0.$$

Equation (1) is an equation not solvable with respect to the highest-order derivative. Such equations are usually called Sobolev type equations since S. L. Sobolev's works were the beginning of the systematic study of such equations [1]. Equation (1) describes the behavior of torsional and longitudinal vibrations of elastic rods [2, 3]. This equation belongs to the class of *pseudohyperbolic equations* introduced in the monograph [4].

We consider the boundary value problem for (1) with the following conditions

$$u\big|_S = 0, \quad \frac{\partial u}{\partial \nu}\bigg|_S = 0, \quad u\big|_{t=0} = 0, \quad D_t u\big|_{t=0} = 0, \qquad (2)$$

where

$$S = \left\{(t,x') \in \overline{Q_T} : \ t \in \left[0,T\right], \ x \in \partial G\right\}.$$

Definition. A function $u(t,x) \in W_2^{1,2}(Q_T)$ such that

$$u_{tx_i} \in L_2(Q_T),\ i = 1,\dots,n, \quad u\big|_{t=0} = 0, \quad u\big|_S = 0, \quad \frac{\partial u}{\partial \nu}\bigg|_S = 0,$$

is called a *generalized solution* to the boundary value problem (1), (2) if the equality holds

$$\int\limits_{Q_T} \Big[- u_t v_t + \sum_{i=1}^{n} \Big[- \alpha_i(x) D^2_{tx_i} u D^2_{tx_i} v + b_i(x) D_{x_i} u D_{x_i} v \Big]$$

$$+ \sum_{i,j=1}^{n} a_{i,j}(x) D^2_{x_i x_j} u D^2_{x_i x_j} v \Big] dxdt = \int\limits_{Q_T} f(t,x) v(t,x) dxdt,$$

for every $v(t,x) \in W_2^{1,2}(Q_T)$ such that $v_{tx_i} \in L_2(Q_T)$, $i = 1,\dots,n$,

$$v\big|_{t=T} = 0, \quad v\big|_S = 0, \quad \frac{\partial v}{\partial \nu}\bigg|_S = 0.$$

Theorem. *Let $f(t,x) \in L_2(Q_T)$, then the boundary value problem (1), (2) has a unique generalized solution $u(t,x) \in W_2^{1,2}(Q_T)$; moreover,*

$$\|u(t,x), W_2^{1,2}(Q_T)\| \le c \|f(t,x), L_2(Q_T)\|,$$

where constant $c > 0$ does not depend on $f(t,x)$.

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ON THE ROBUST STABILITY OF STATIONARY SOLUTIONS TO A CLASS OF MATHIEU-TYPE EQUATIONS

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We consider the stability of stationary solutions to quasilinear equation with parameters

$$y'' + \alpha\mu y' + (\beta\mu^2 + \mu\varphi(t))f(y) = 0,$$

where $\alpha, \beta > 0$, $\varphi(t)$ is a continuous T-periodic function with zero mean value over the period, $f(y)$ is a smooth function, and $\mu > 0$ is a small parameter.

We establish conditions for perturbations of the coefficients of the equation under which the zero solution is asymptotically stable. Estimates for attraction sets of the zero solution and estimates of the stabilization rate of solutions at infinity are obtained. Using these results, theorems on the robust stability of stationary solutions are proven.

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PROPERTIES OF SOLUTIONS TO ONE CLASS OF SYSTEMS OF NONLINEAR DIFFERENTIAL EQUATIONS

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In this paper we consider the Cauchy problem for the system of nonlinear ordinary differential equations

$$\begin{cases} \frac{dz_1}{dt} = g(t,z_n) - \frac{n-1}{\tau_1}\frac{z_1}{1+\rho_1(n-1)^{-\gamma_1}} + \frac{n-1}{\tau_2}\frac{z_2}{1+\rho_2(n-1)^{-\gamma_2}}, \qquad t>0, \\ \frac{dz_j}{dt} = \frac{n-1}{\tau_1}\frac{z_{j-1}}{1+\rho_{j-1}(n-1)^{-\gamma_{j-1}}} - \left(\frac{n-1}{\tau_1}+\frac{n-1}{\tau_2}\right)\frac{z_j}{1+\rho_j(n-1)^{-\gamma_j}} \\ \qquad + \frac{n-1}{\tau_2}\frac{z_{j+1}}{1+\rho_{j+1}(n-1)^{-\gamma_{j+1}}}, \quad j=2,\dots,n-2, \\ \frac{dz_{n-1}}{dt} = \frac{n-1}{\tau_1}\frac{z_{n-2}}{1+\rho_{n-2}(n-1)^{-\gamma_{n-2}}} - \left(\frac{n-1}{\tau_1}+\frac{n-1}{\tau_2}\right)\frac{z_{n-1}}{1+\rho_{n-1}(n-1)^{-\gamma_{n-1}}}, \\ \frac{dz_n}{dt} = -\theta z_n + \frac{n-1}{\tau_1}\frac{z_{n-1}}{1+\rho_{n-1}(n-1)^{-\gamma_{n-1}}}, \\ z|_{t=0} = z_0, \end{cases} \tag{1}$$

where $\theta > 0$, $0 \le \rho_j < \rho$, $\gamma_j > \gamma > 1$, $\tau_2 > \tau_1 > 0$. Here $g(t,z) \in C(\overline{\mathbb{R}}^2_+)$ is a bounded function and satisfies Lipschitz condition with respect to the second argument. Such systems appear in multistage synthesis models (see, for example, [1]), where $z_n(t)$ describes consentration of the final synthesis product. Since number of stages n can be very large, then a problem of "large dimension" occurs when finding $z_n(t)$.

In this paper we study properties of solutions to (1) for $n \gg 1$ and $\tau_2 \gg 1$. Using methods proposed by G. V. Demidenko (see, for instance, [2, 3] we prove closeness of $z_n(t)$ for $n \gg 1$ and the solution $y(t)$ to the following delay differential equation

$$\frac{dy}{dt} = -\theta y + g(t-\tau, y(t-\tau)),$$

where $\tau = \frac{\tau_1\tau_2}{\tau_2-\tau_1}$. We prove that the components of the solution to (1) can be approximated by the components of the solution to the following Cauchy problem for $\tau_2 \gg 1$

$$\begin{cases} \frac{dx_1}{dt} = g(t, x_n) - \frac{n-1}{\tau_1(1+\rho_1(n-1)^{-\gamma_1})} x_1, & t > 0, \\ \frac{dx_j}{dt} = \frac{n-1}{\tau_1(1+\rho_{j-1}(n-1)^{-\gamma_{j-1}})} x_{j-1} - \frac{n-1}{\tau_1(1+\rho_j(n-1)^{-\gamma_j})} x_j, & j = 2, \dots, n-1, \\ \frac{dx_n}{dt} = -\theta x_n + \frac{n-1}{\tau_1(1+\rho_{n-1}(n-1)^{-\gamma_{n-1}})} x_{n-1}, \\ x|_{t=0} = z_0. \end{cases}$$

Estimates characterizing convergence rates as $n \to \infty$ and $\tau_2 \to \infty$ were obtained in [4].

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REDUCED-ORDER OBSERVER-BASED CONTROLLER DESIGN FOR QUASI-ONE-SIDED LIPSCHITZ NONLINEAR SYSTEMS WITH TIME-DELAY

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This paper deals with the problem of reduced-order observer-based controller design for a class of nonlinear time-delay systems. The systems described as

$$\begin{cases} \dot{x} = Ax + A_\tau x_\tau + Bu + \phi(x, x_\tau), \\ y = Cx, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^p$ is the output vector, $u \in \mathbb{R}^m$ is the control input vector, $A, A_\tau \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$ are the constant matrices of appropriate dimensions, $x_\tau = x(t-\tau)$ and τ is the positive constant time-delay; $\phi(x, x_\tau)$ is a nonlinear function with respect to x, x_τ.

We employ the partitions of the matrices as follows:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad A_\tau = \begin{pmatrix} A_{\tau 11} & A_{\tau 12} \\ A_{\tau 21} & A_{\tau 22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

where $A_{11}, A_{\tau 11} \in \mathbb{R}^{p\times p}$, $A_{12}, A_{\tau 12} \in \mathbb{R}^{p\times(n-p)}$, $A_{21}, A_{\tau 21} \in \mathbb{R}^{(n-p)\times p}$, $A_{22}, A_{\tau 22} \in \mathbb{R}^{(n-p)\times(n-p)}$, $B_1 \in \mathbb{R}^{p\times m}$ and $B_2 \in \mathbb{R}^{(n-p)\times m}$.

The sufficient conditions for the existence of the reduced-order observer of nonlinear time-delay systems (1) is proposed.

Theorem. *Consider nonlinear time-delay system* (1) *with quasi-one-sided Lipschitz condition. If there exists some matrix $Q > 0$, gain matrices L and L_τ can be chosen such that the following matrix inequality*

$$\begin{pmatrix} (A-LC)^T P + P(A-LC) + Q + 2M & P(A_\tau - L_\tau C) \\ * & -Q + 2N \end{pmatrix} < 0$$

holds, where the positive-definite matrix P and the symmetric matrices M, N satisfy the quasi-one-sided Lipschitz condition, then there exists a reduced-order observer for nonlinear system (1) *given by*

$$\begin{cases} \dot{\hat{z}}_2 = (A_{22} + FA_{12})\hat{z}_2 + (A_{\tau 22} + FA_{\tau 12})\hat{z}_{\tau 2} + (FA_{11} - FA_{12}F \\ + A_{21} - A_{22}F)y + (FA_{\tau 11} - FA_{\tau 12}F + A_{\tau 21} - A_{\tau 22}F)y_\tau \\ + (FB_1 + B_2)u + (F, I_{n-p})\phi\left(\begin{pmatrix} y \\ \hat{z}_2 - Fy \end{pmatrix}, \begin{pmatrix} y_\tau \\ \hat{z}_{\tau 2} - Fy_\tau \end{pmatrix}\right), \\ \hat{z}_1 = \hat{x}_1 = y, \\ \hat{x}_2 = \hat{z}_2 - Fy, \end{cases} \tag{2}$$

where $F = P_3^{-1}P_2^T \in \mathbb{R}^{(n-p)\times p}$.

Subsequently, a state feedback controller is designed, the sufficient conditions that the zero solution of the closed-loop system is asymptotically stable is proposed. Then, a reduced-order observer-based controller is proposed for stabilization of system (1).

Combining the controller

$$u = -K\hat{x} - K_\tau \hat{x}_\tau$$

with reduced-order observer (2), by nonsingular state transformation and constructing series Lyapunov–Krasovskii functional

$$V(x, e_2) = b\,x^T P_0 x + b\int_{t-\tau}^{t} x^T(s)Q_0 x(s)ds + e_2^T P_3 e_2 + \int_{t-\tau}^{t} e_2^T(s)Q_3 e_2(s)ds,$$

it is shown that the separation principle holds for stabilization of the systems based on the reduced-order observer-based controller.

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MODELING THE DYNAMICS OF HEALTH-RELATED QUALITY OF LIFE

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Specialists in preventive and clinical medicine are often faced with the need to make decisions on the management of individual components of the dynamic of health-related quality of life (HRQoL). Judgments about the quality of any object have the property of variability depending on time, therefore, in system dynamic models, the overall picture of HRQoL should be considered over time. The analysis of the dynamic characteristics forming the integral assessment of HRQoL is not sufficiently developed according to modern available domestic and foreign literature. So, modeling the dynamics of HRQoL is relevant.

For modeling, we took a structural model in the form of a graph-tree with three levels of hierarchy, built on the basis of information from the nonspecific SF-36 questionnaire, approved by the WHO as a tool for assessing HRQoL [1]. Elements of HRQoL as a system are distributed in the model into levels, at each of which square matrices of paired comparisons are compiled, reflecting qualitative judgments of experts about the dynamics of HRQoL, which are then converted into quantitative ones using of hierarchies analysis method (HAM) by T. Saaty [2]. Classes of functions describing the dynamics of HRQoL criteria were identified. The priority (weight) vectors $w(t)$ are eigenvectors of these matrices and found from solving linear equation of the form

$$A(t)w(t) = \lambda_{max}(t)w(t)$$

for each fixed t from some segment $[t_0, t_0 + T]$.

The problems of HRQoL research are related to the multidimensionality of scales and their heterogeneity. The first problem is solved by using the method of nested linear convolution, which has the form

$$J = w_{21} \sum_{i=1}^{4} w_{3i} x_{3i} + w_{22} \sum_{i=5}^{8} w_{3i} x_{3i}, \tag{1}$$

where $(x_{31}, \ldots, x_{38})$ is a vector of the lower level of the hierarchy, representing the survey results (indicators) in points from 0 to 100 for scales of the HROoL. Priority vectors of the second and third levels of the hierarchy are constructed using HAM and represent the weights of these scales in fractions of 1.

Let us denote by $w = (w_1, \ldots, w_8)$ the vector of linear convolution coefficients (1), and by $x = (x_1, \ldots, x_8)$ the vector of indicators of HRQoL scales of the third (lower) level of the hierarchy. Then integral indicator of HROoL has a form $I(w, x) = J(w, x)/100$ and its maximum equal to 1 is achieved at $x_{3i} = 100$ for all $i = 1, \ldots, 8$.

In a real situation research $x(t)$ cannot be carried out continuously. Therefore, it is necessary to select a time period and discretize the task of studying HRQoL. In this case, it may be useful to idealize the model in the form of ordinary differential equation

$$\dot{x} = f(t, w(t), x, u(t, x)) \tag{2}$$

and to select the difference equations. We consider the difference equation of Euler method

$$x(t_k + h) = x(t_k) + hf(t_k, w(t_k), x(t_k), u(t_k, x(t_k))) \tag{3}$$

in vector form, where $h = T/N$, $k = 0, \ldots, N-1$.

The vector-function $f(t, w, x, u) = (f_1, \ldots, f_8)$ is to be determined based on the problems of the research. In particular, this may be the problem of optimal management resources, the task of the fastest growth of the integral indicator of HRQoL, the task of finding the equilibrium position for equations (2) and (3) and its stabilization.

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QUASILINEAR EQUATIONS WITH DISTRIBUTED GERASIMOV–CAPUTO DERIVATIVES

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Let $\mathcal{Z}$ be Banach space, $\mathcal{C}l(\mathcal{Z})$ be a set of linear closed operators, densely defined in $\mathcal{Z}$, $S_{\theta,a} := \{\mu \in \mathbb{C} : |\arg(\mu - a)| < \theta,\ \mu \neq a\}$ for $\theta \in [\pi/2, \pi]$, $a \in \mathbb{R}$. Define a class $\mathcal{A}_W(\theta_0, a_0)$ [1] of all operators $A \in \mathcal{C}l(\mathcal{Z})$, such that:

(i) there exist $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, such that $W(\lambda) \in \rho(A)$ for every $\lambda \in S_{\theta_0,a_0}$;

(ii) for every $\theta \in (\pi/2, \theta_0)$, $a > a_0$, there exists $K(\theta, a) > 0$, such that for all $\lambda \in S_{\theta,a}$

$$\|(W(\lambda)I - A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{|\lambda| K(\theta, a)}{|W(\lambda)||\lambda - a|}.$$

Let $b < c$, $m - 1 < c \leq m \in \mathbb{N}$, $b_l < c_l$, $m_l - 1 < c_l \leq m_l \in \mathbb{Z}$, $c_1 \leq c_2 \leq \cdots \leq c_n < c$, $\mu \in BV((b, c]; \mathbb{C})$, $\mu_l \in BV((b_l, c_l]; \mathbb{C})$, $l = 1, 2, \ldots, n$, $T > t_0$. A solution on a segment $[t_0, T]$ of the Cauchy problem

$$D^k z(t_0) = z_k, \quad k = 0, 1, \ldots, m - 1, \tag{1}$$

for the equation

$$\int_b^c D^\alpha z(t) d\mu(\alpha) = Az(t) + B\left(t, \int_{b_1}^{c_1} D^\alpha z(t) d\mu_1(\alpha), \ldots, \int_{b_n}^{c_n} D^\alpha z(t) d\mu_n(\alpha)\right), \tag{2}$$

is a function $z \in C^{m-1}([t_0, T]; \mathcal{Z}) \cap C((t_0, T]; D_A)$, such that $\int_b^c D^\alpha z(t) d\mu(\alpha) \in C((t_0, T]; \mathcal{Z})$, $\int_{b_l}^{c_l} D^\alpha z(t) d\mu_l(\alpha) \in C([t_0, T]; \mathcal{Z})$, $l = 1, 2, \ldots, n$, and equalities (1) and (2) for $t \in (t_0, T]$ are fulfilled.

Theorem [2]. *Let $m - 1 < c \leq m \in \mathbb{N}$, $b < c$, $\mu \in BV((b, c]; \mathbb{C})$, c be a variation point of the measure $d\mu(\alpha)$, $n \in \mathbb{N}$, $c_1 \leq c_2 \leq \cdots \leq c_n < c$, $b_l < c_l$, $\mu_l \in BV((b_l, c_l]; \mathbb{C})$, c_l be a variation point of the measure $d\mu_l(\alpha)$, $l = 1, 2, \ldots, n$, $A \in \mathcal{A}_W(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$, $z_k \in D_A$, $k =$*

$0, 1, \ldots, m-1$, *a mapping* $B \in C([t_0, T] \times \mathcal{Z}^n; \mathcal{Z})$ *be Lipschitz continuous. Then, problem* (1), (2) *have a unique solution on the segment* $[t_0, T]$.

Consider a bounded region $\Omega \subset \mathbb{R}^d$ with a smooth boundary $\partial\Omega$, $\beta, \gamma, \nu \in \mathbb{R}$, $c \in (1, 2)$, $b < c$, $\alpha_1 < \alpha_2 < \cdots < \alpha_n \leq c$, $\omega_k \in \mathbb{R} \setminus \{0\}$, $k = 1, 2, \ldots, n$, $\omega \in C([b, c]; \mathbb{R})$; if $\alpha_n < c$, then $\omega(c) \neq 0$ in a some left vicinity of c; $\beta_l < c$, $b_l < c_l < c$, $\mu_l \in BV((b_l, c_l]; \mathbb{R})$, $l = 1, 2$. Consider the initial-boundary value problem

$$u(s,0) = u_0(s), \quad v(s,0) = v_0(s), \quad s \in \Omega,$$

$$\frac{\partial u}{\partial t}(s,0) = u_1(s), \quad \frac{\partial v}{\partial t}(s,0) = v_1(s), \quad s \in \Omega,$$

$$u(s,t) = v(s,t) = 0, \quad (s,t) \in \partial\Omega \times (0, T],$$

for the nonlinear system of equations in $\Omega \times (0, T]$

$$\sum_{k=1}^{n} \omega_k D_t^{\alpha_k} u(s,t) + \int_b^c \omega(\alpha) D_t^{\alpha} u(s,t) d\alpha = \Delta u(s,t) - \Delta v(s,t)$$

$$+F_1 \left(s, D^{\beta_1} u(s,t), \int_{b_1}^{c_1} D^{\alpha} v(s,t) d\mu_1(\alpha) \right),$$

$$\sum_{k=1}^{n} \omega_k D_t^{\alpha_k} v(s,t) + \int_b^c \omega(\alpha) D_t^{\alpha} v(s,t) d\alpha = \nu \Delta v(s,t) + \beta u(s,t) + \gamma v(s,t)$$

$$+F_2 \left(s, D^{\beta_2} v(s,t), \int_{b_2}^{c_2} D^{\alpha} u(s,t) d\mu_2(\alpha) \right).$$

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METHOD OF LIMIT DIFFERENTIAL INCLUSIONS FOR STUDYING THE ASYMPTOTIC BEHAVIOR OF NON-AUTONOMOUS SYSTEMS

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We are developing methods for studying the asymptotic behavior of non-autonomous systems presented in the form of differential inclusions, discontinuous systems and systems with delay. The received results carry the form of generalizations of the LaSalle's principle of invariance. Within the framework of the Lyapunov's direct method, it is assumed that the derivative of the Lyapunov function is non-positive. For autonomous systems, the following conclusion can be drawn from this: ω-limit sets of solutions belong to the largest invariant subset of the set of zeros of the derivative of the Lyapunov's function of these systems.

Difficulties in studying non-autonomous systems are associated with the lack of invariance properties of ω-limit sets for solutions of such systems, as well as with the description of the set of zeros of the derivative of Lyapunov functions. Attempts to overcome these difficulties led to the emergence of the concept of limit differential equations. The method of limit equations in non-autonomous systems goes back to the works of G. R. Sell [1] and Z. Arshtein [2] on the topological dynamics of non-autonomous differential equations.

Extending the method of limit equations to wider classes of systems raises a fundamental question about the structure of limit equations. We solve this problem by passing to limit differential inclusions.

Let R^n be a n-dimensional vector space, $conv\, R^n$ be the collection of all non-empty compact convex subsets of R^n. We consider a non-autonomous differential inclusion

$$\dot{x} \in F(t,x), \quad x(t_0) = x_0, \tag{1}$$

where $F : R^1 \times R^n \to conv\, R^n$ is such that for every fixed x the multivalued mapping $t \to F(t,x)$ has a measurable selector. In particular, this takes place in the theory of discontinuous systems when determining a solution in the sense of A. F. Filippov. Here, to construct limit maps it is not possible to use any theorems and facts of mathematical and multivalued analysis on the convergence of functional sequences.

For further we will also assume that multivalued mapping $x \to F(t,x)$ be upper semicontinuous and limited on every set $R^1 \times K$, where $K \subset R^n$ is a compact set.

Let us formulate one of the theorems based on the study of the limit differential inclusion

$$\dot{x} \in F^*(x) \stackrel{\text{def}}{=} \cap_{t\geq 0} \overline{\text{co}}\, F(t,x), \tag{2}$$

where $\overline{\text{co}}$ is the sign of the convex closed hull of the set.

Definition. A set $D \subset R^n$ is semi-invariant, if for any point $y \in D$ there exists a solution $y(t)$ to inclusion (2) such that $y(0) = y$ and $y(t) \in D$ for all $t \geq 0$.

Theorem. *Let $w(t,x)$ be a bounded function of Caratheodory type and the inequality*

$$\dot{V}^+(t,x) \stackrel{\text{def}}{=} \sup_{y\in F(t,x)} (\langle \nabla_x V, y\rangle + V_t) \leq w(t,x)$$

holds, where $\nabla_x V$ is the gradient of $V(t,x)$ with respect to the variable x, V_t is the partial derivative with respect to t and $\langle \cdot, \cdot \rangle$ is scalar product. Then the ω-limit set of any bounded solution $x(t)$ of the inclusion (1) *belongs to the largest semi-invariant subset of the set*

$$E(\alpha = 0) \stackrel{\text{def}}{=} \{x \in R^n : \alpha(x) = 0\},$$

where $\alpha(x)$ is lower limit of function $t \to w(t,x)$ under condition $t \to +\infty$.

Based on this Theorem, in article [3] it was studied the asymptotic behavior of mechanical systems with dry friction at the Lagrange form.

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DEGENERATE INTEGRO-DIFFERENTIAL EQUATIONS WITH BOUNDED OPERATORS IN BANACH SPACES AND THEIR APPLICATIONS

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The work investigates integro-differential equations in Banach spaces with operators, which are a composition of convolution and differentiation operators. The conditions of the unique solvability of the Cauchy type problem for a degenerate integro-differential equation of the Riemann–Liouville type are obtained. Examples of integro-differential operators, which are various fractional derivatives, are considered.

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, $M : D_M \to \mathcal{Y}$ is a linear closed operator, $\overline{D}_M = \mathcal{X}$, L is a linear bounded operator, $\ker L \neq \{0\}$. If $\{\mu \in \mathbb{C} : |\mu| > a\} \subset \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathcal{Y};\mathcal{X})\}$, then there are projections

$$P = \frac{1}{2\pi i} \int\limits_{|\mu|=r} (\mu L - M)^{-1} L d\mu, \quad Q = \frac{1}{2\pi i} \int\limits_{|\mu|=r} L(\mu L - M)^{-1} d\mu, \quad r > a,$$

on $\mathcal{X}$ and $\mathcal{Y}$, respectively. Put $\mathcal{X}^0 := \ker P$, $\mathcal{X}^1 := \mathrm{im} P$, $\mathcal{Y}^0 := \ker Q$, $\mathcal{Y}^1 := \mathrm{im} Q$, $L_k := L|_{\mathcal{X}^k}$, $M_k := M|_{\mathcal{X}^k \cap D_M}$, $k = 0, 1$. It is known [1], that $L_k, M_k : \mathcal{X}^k \to \mathcal{Y}^k$, $k = 0, 1$, there is $M_0^{-1} \in \mathcal{L}(\mathcal{Y}^0;\mathcal{X}^0)$. Denote $G := M_0^{-1} L_0$; under the condition $G^p \neq 0$, $G^{p+1} = 0$ for some $p \in \mathbb{N} \cup \{0\}$ operator M is called (L, p)-bounded. Define the integro-differential operator $(D^{m,K} x)(t) := D^m \int_0^t K(t-s)x(s)ds$, where D^m is a usual derivative of the order m.

Theorem [2]. *Let an operator M be (L, p)-bounded, $K \in C(\mathbb{R}_+;\mathcal{L}(\mathcal{X}))$, there exists the Laplace transform $\widehat{K}$, which be single-valued analytic operator-function in $\Omega_{R_0} := \{\mu \in \mathbb{C} : |\mu| > R_0, |\arg \mu| < \pi\}$ for some $R_0 > 0$ and condition $\|\widehat{K}(\lambda)\|_{\mathcal{L}(\mathcal{X})} > c|\lambda|^{\chi-1}$ in Ω_{R_0} for some $\chi, c > 0$ holds. Suppose that for all $\lambda \in \Omega_{R_0}$ there exists $\widehat{K}(\lambda)^{-1} \in \mathcal{L}(\mathcal{X})$, $g \in C((0,T];\mathcal{Y}) \cap L_1(0,T;\mathcal{Y})$, for $l = 0, 1, \ldots, p$*

$$(D^{m,K} G)^l M_0^{-1}(I - Q)g, \ D^{m,K}(D^{m,K} G)^l M_0^{-1}(I - Q)g \in C((0,T];\mathcal{X}),$$

$x_k \in \mathcal{X}^1$, $k = 0, 1, \ldots, m-1$. *Then there exists a unique solution to problem* $D^{k,K}(Px)(0) = x_k$, $k = 0, 1, \ldots, m-1$, $LD^{m,K}x(t) = Mx(t) + g(t)$, $t \in (0, T]$, *it has the form*

$$x(t) = \sum_{k=0}^{m-1} U_k(t)x_k + \int_0^t U_{m-1}(t-s)L_1^{-1}Qg(s)ds$$

$$-\sum_{l=0}^{p}(D^{m,K}G)^l M_0^{-1}(I-Q)g(t),$$

where for $k = 0, 1, \ldots, m-1$

$$U_k(t) = \frac{1}{2\pi i}\int_{\partial\Omega_R}(\lambda^m \widehat{K}(\lambda) - L_1^{-1}M_1)^{-1}\lambda^{m-1-k}e^{\lambda t}d\lambda, \quad t > 0.$$

Example 1. Take $m-1 < \alpha \leq m \in \mathbb{N}$, $K_\alpha(s) := \frac{s^{\alpha-1}}{\Gamma(\alpha)}I$, then $D^{m,K_{m-\alpha}}$ is the operator of the fractional Riemann–Liouville differentiation of the order α.

Example 2. Take $a \in \mathbb{R}$, $\alpha > 0$, $\beta \in (0, 1)$, $K(s) = s^{-\beta}E_{\alpha,1-\beta}(as^\alpha)I$, where $E_{\alpha,\delta}(z) = \sum\limits_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\delta)}$ is the Mittag–Leffler function, then

$$D^{m,K}x(t) = D^m\int_0^t (t-s)^{-\beta}E_{\alpha,1-\beta}(a(t-s)^\alpha)x(s)ds$$

is the fractional derivative of Prabhakar [3].

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MATHEMATICAL AND NUMERICAL MODELING OF THE PLURIPOTENCY GENE NETWORK DYNAMICS

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Following [1], we consider 10-D dynamical system as a model of one pluripotency gene network functioning:

$$\frac{du_1}{dt} = F_1(A, u_3, v_2) - k_1 u_1; \quad \frac{du_2}{dt} = F_2(A, u_3, v_2) - k_2 u_2;$$

$$\frac{du_3}{dt} = \mu_3 u_4 u_6 - (k_3 + \alpha_3) u_3; \quad \frac{du_4}{dt} = \alpha_3 u_3 + \mu_4 u_5 - (k_4 + \mu_3 u_6) u_4;$$

$$\frac{du_5}{dt} = \alpha_5 u_1 - (k_5 + \mu_4) u_5; \quad \frac{du_6}{dt} = \alpha_3 u_3 + \mu_6 u_7 - (k_6 + \mu_3 u_4) u_6;$$

$$\frac{du_7}{dt} = \alpha_7 u_2 - (k_7 + \mu_6) u_7; \quad \frac{dv_1}{dt} = F_3(u_3, v_2) - k_8 v_1;$$

$$\frac{dv_2}{dt} = \alpha_9 v_3 - k_9 v_2; \qquad \frac{dv_3}{dt} = \alpha_{10} v_1 - (\alpha_9 + k_{10}) v_3. \tag{1}$$

Some feedbacks here are described by the functions

$$F_1(A, u_3, v_2) = \frac{a_2 + a_3 A + a_4 u_3^m + q_1 v_2^s}{1 + a_7 A + a_8 u_3^m + q_2 v_2^s};$$

$$F_2(A, u_3, v_2) = \frac{a_{11} + a_{12} A + a_{13} u_3^n + q_3 v_2^r}{1 + a_{16} A + a_{17} u_3^n + q_4 v_2^r};$$

$$F_3(u_3, v_2) := \frac{1 + b_3 u_3^\ell}{1 + b_6 u_3^\ell + (1 + b_8 u_3^\ell) v_2^h}.$$

The non-negative variables u_j, v_k denote concentrations of the components of the gene network, A is an external signal. The biological interpretations are exposed in [1, 2]. In our studies, in contrast with [1, 2], we do not fix the values of the parameters in the system (1); we just impose here some additional symmetry conditions which are satisfied in [1]:

$F_1(A, u_3, v_2) \equiv F_2(A, u_3, v_2)$, $\alpha_5 = \alpha_7$, $k_1 = k_2$, $k_4 = k_6$, $k_5 = k_7$.

Under these assumptions, we find conditions of existence of several equilibrium points of the system (1) and describe conditions of their stability. These points correspond to different states of the stem cells, see [1, 2].

Numerical illustrating experiments with trajectories of the system (1) are realized in a cloud web service: https://colab.research.google.com/drive/1GicSvb_e95dRe_U451_r8F-1Fr97QLu7?usp=sharing
This client-server application was elaborated using approaches described in [3].

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THE WORKS OF THE S. K. GODUNOV SEMINAR ON HYPERBOLIC EQUATIONS

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To Sergey Konstatinovich Godunov on his memory

In the mid 1970s, at Novosibirsk State University, the S. K. Godunov seminar on hyperbolic equations started its work. The report describes the works of the participants on hyperbolic equations. The main interest was concentrated around two problems. The first is the reduction of a high-order Petrovskiĭ hyperbolic equation to a first-order Friedrichs hyperbolic symmetric system. The second problem is that if a boundary value problem is posed for a hyperbolic equation, then it is required to reduce it to a symmetric system so that the posed boundary condition be dissipative.

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STUDY OF LINEAR STABILITY FOR CYLINDRICALLY SYMMETRICAL STATES OF DYNAMIC EQUILIBRIUM OF TWO-COMPONENT VLASOV–POISSON PLASMA

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In the electrostatic approximation, when the electric field of electrons and ions is self-consistent, the plasma dynamics is described by the kinetic Vlasov–Poisson equations [1]. In this case, such equations are used to study the collisionless motion of electrons, which interact with each other through the Coulomb repulsive forces, against the background of a homogeneous distribution of ions in the whole physical continuum.

The aim of this research is to prove an absolute instability for the exact stationary cylindrically symmetrical solutions to kinetic Vlasov–Poisson equations by the direct Lyapunov method [2] with respect to the small cylindrically symmetrical perturbations. The results of this study are important for solving the problem of controlled thermonuclear fusion.

To achieve such goal, the hydrodynamic substitution of independent variables is performed so that kinetic Vlasov–Poisson equations are transformed to an infinite system of cylindrically symmetrical equations similar to the equations of isentropic flows of compressible fluid medium in the vortex shallow water and the Boussinesq approximations [2]. The new defining equations have the exact stationary solutions that are equivalent to the exact stationary cylindrically symmetrical solutions to kinetic Vlasov–Poisson equations. Then these defining equations are linearized near their exact stationary solutions. The a priori exponential estimate from below is constructed for a subclass of small cylindrically symmetrical perturbations of exact stationary solutions to new defining equations, which grow over time and are described by the field of Lagrangian displacements [2]. Since the estimate is obtained for any exact stationary solutions to new defining equations, it proves precisely the absolute linear instability of these solutions with regard to the small cylindrically symmetrical perturbations from the subclass mentioned above. Thus, the Newcomb–Gardner–Rosenbluth sufficient condition [3] for linear stability of the exact stationary cylindrically symmetrical solutions to kinetic Vlasov–Poisson equations is conversed, and its formal character is revealed. Also, the sufficient conditions for linear practical instability of the exact stationary solutions to new defining equations are found, and their constructive nature is discovered. At last, the results of this research are consistent with the well-known Earnshaw theorem [1, 3] on instability in electrostatics and extend the scope of its applicability from classical mechanics to statistical one.

As for the significance of these results, they can be used to study the adequacy of mathematical models for plasma to the physical phenomena which the models describe. Furthermore, the results obtained here can be applied to the development and subsequent operation of devices designed to perform the controlled thermonuclear fusion. In order for a plasma confinement device to operate reliably, it needs for us to ensure the practical stability of its dynamic equilibrium states with respect to all acceptable perturbations. In particular, these equilibrium states should be robust in a practical sense for small cylindrically symmetrical perturbations. This can be achieved by creation of numerical and physical models, which correspond to the linearized initial-boundary value problem under investigation, with control the sufficient conditions for linear practical instability at some reference time points. In constructing these models, the main focus should be on ensuring that the sufficient conditions for linear practical instability are not met at the expense of those or other known external influences on small cylindrically symmetrical perturbations growing with time (for example, by virtue of suitable setting of initial conditions). In consequence, the operation reliability of the device for plasma confinement in working mode will be guaranteed.

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STUDY OF INSTABILITY FOR THREE-DIMENSIONAL DYNAMIC EQUILIBRIUM STATES OF SELF-GRAVITATING VLASOV–POISSON GAS

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The Vlasov–Poisson model of boundless collisionless gas of neutral particles in a self-consistent gravitational field continues to be one of the basic models of modern astrophysics. This is due to simplicity, clarity, and obvious effectiveness of the model in describing large-scale processes in the Universe.

In this work, we consider the spatial motions of boundless collisionless self-gravitating Vlasov–Poisson gas of neutral particles in a three-dimensional Cartesian coordinate system:

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \varphi}{\partial x_i}\frac{\partial f}{\partial v_i} = 0, \quad \frac{\partial^2 \varphi}{\partial x_i^2} = 4\pi \left(\int\limits_{\mathbb{R}^3} f(\mathbf{x},\mathbf{v},t)d\mathbf{v} - n_g \right), \qquad (1)$$

$$i = 1,2,3; \quad f = f(\mathbf{x},\mathbf{v},t) \geq 0; \quad f(\mathbf{x},\mathbf{v},0) = f_0(\mathbf{x},\mathbf{v}).$$

Here, f is the distribution function of neutral particles (for reasons of convenience, their masses are assumed to be the same and equal to unity); t is time; $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are coordinates and velocities of neutral particles; $\varphi(\mathbf{x}, t)$ is the potential of a self-consistent gravitational field; $d\mathbf{v} \equiv dv_1 dv_2 dv_3$ is the differential volume element in the velocity space; $4\pi n_g \equiv const > 0$ is the gas particles density in some three-dimensional static state of global thermodynamic equilibrium; $f_0(\mathbf{x},\mathbf{v})$ is the initial data for function f. We suppose that the distribution function f asymptotically approaches zero as $|\mathbf{v}| \to \infty$, and this function along with the potential φ are periodic in argument $\mathbf{x}$ or asymptotically approach zero as $|\mathbf{x}| \to \infty$ too. Summation is performed on repeating lower index i.

It is assumed that the mixed problem (1) has the following exact stationary solutions:

$$f = f^0(\mathbf{v}) \geq 0, \quad \varphi = \varphi^0 \equiv const; \quad \int\limits_{\mathbb{R}^3} f^0(\mathbf{v}) d\mathbf{v} = n_g. \tag{2}$$

The aim of this work is to prove an absolute linear instability for the spatial states (2) of dynamic equilibrium of boundless collisionless self-gravitating Vlasov–Poisson gas with respect to small three-dimensional perturbations.

For that purpose, a transition from kinetic equations (1) to an infinite system of relations similar to the equations of isentropic flow of a compressible fluid medium in the "vortex shallow water" and Boussinesq approximations was carried out. In the course of instability proof, the well-known sufficient Newcomb–Gardner–Rosenbluth condition for stability of dynamic equilibrium states (2) with respect to one incomplete unclosed subclass of small spatial perturbations was conversed. Also, a linear ordinary differential second-order inequality with constant coefficients for the Lyapunov functional was obtained. An a priori exponential lower estimate for the growth of small three-dimensional perturbations follows from this inequality when the sufficient conditions for linear practical instability of the considered dynamic equilibrium states are satisfied. Since the obtained estimate was deduced without any additional restrictions on the equilibrium states under study, then the absolute linear instability of spatial states (2) of dynamic equilibrium of the Vlasov–Poisson gas with respect to small three-dimensional perturbations was thereby proved.

The results of this work are fully consistent with the classical Earnshaw instability theorem. Moreover, the area of applicability for the Earnshaw theorem is expanded now from electrostatics to kinetics, namely, to the boundless collisionless self-gravitating Vlasov–Poisson gas of neutral particles.

Constructiveness is inherent in the sufficient conditions for linear practical instability established here, which allows them to be used as a testing or control mechanism for conducting physical experiments and performing numerical calculations.

As a means of confirming the results obtained, a series of analytical examples of the considered dynamic equilibrium states are constructed along with small spatial perturbations superimposed on them that grow in time as identified by the found estimate.

STABILIZATION OF HIGH-ORDER DELAY SYSTEMS: A FREQUENCY-DOMAIN METHOD

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Problem Description: Consider linear high-order delay systems

$$x^{(n)}(t)+\sum_{l=1}^{n}[A_l x^{(n-l)}(t)+\sum_{j=1}^{m}D_{l,j}x^{(n-l)}(t-\tau_j)]=Bu(t), \qquad (1)$$

where $A_l, D_{l,j} \in \mathbf{R}^{d\times d}$ for $l=1,\dots,n$, $j=1,\dots,m$, $B\in\mathbf{R}^{d\times p}$ are constants, delays $\tau_j>0$, state $x(t)\in\mathbf{R}^d$, control $u(t)\in\mathbf{R}^p$, and the indexes on x denote derivatives with respect to the independent variable t.

Let a controller be

$$u(t)=-\{\sum_{l=1}^{n}[K_l x^{(n-l)}(t)+\sum_{j=1}^{m}F_{l,j}x^{(n-l)}(t-\tau_j)]\},$$

where feedback gain matrices $K_l, F_{l,j}\in\mathbf{R}^{p\times d}$. The closed-loop system

$$x^{(n)}(t)+\sum_{l=1}^{n}[\tilde{A}_l x^{(n-l)}(t)+\sum_{j=1}^{m}\tilde{D}_{l,j}x^{(n-l)}(t-\tau_j)]=0, \qquad (2)$$

where $\tilde{A}_l=A_l+BK_l$ and $\tilde{D}_{l,j}=D_{l,j}+BF_{l,j}$. Our aim is to seek feedback gain matrices K_l and $F_{l,j}$ such that system (2) is asymptotically stable.

Main Results: For system (2), let $y_1(t)=x(t), y_2(t)=x^{(1)}(t)=\dot{y}_1(t),\dots,y_n(t)=x^{(n-1)}(t)=\dot{y}_{n-1}(t)$. System (2) can be written as the first-order system [1]

$$\dot{Y}(t)=\mathcal{A}Y(t)+\sum_{j=1}^{m}\mathcal{D}_jY(t-\tau_j),$$

where $Y(t)=[y_1(t),y_2(t),\dots,y_n(t)]^T\in\mathbf{R}^{nd}$, $\mathcal{A}$ and $\mathcal{D}_j\in\mathbf{R}^{nd\times nd}$ are determined by $\tilde{A}_{n-l}$ and $\tilde{D}_{n-l,j}$, respectively.

Let $\vec{K}$ denote the compound matrix $[K_1, K_2, \ldots, K_n, F_{1,1}, \ldots, F_{n,m}]$. The fundamental matrix of system (2) is denoted by $G[\vec{K}, t] \in \mathbf{R}^{nd\times nd}$, which satisfies

$$\dot{G}[\vec{K}, t] = \mathcal{A}G[\vec{K}, t] + \sum_{j=1}^{m} \mathcal{D}_j G[\vec{K}, t - \tau_j]$$

with $G[\vec{K}, t] = 0$, $t < 0$ and $G[\vec{K}, 0] = I$.

$$\text{Let} \quad G[\vec{K}, t] = \begin{bmatrix} G_{1,1}(t) & G_{1,2}(t) & \ldots & G_{1,n}(t) \\ G_{2,1}(t) & G_{2,2}(t) & \ldots & G_{2,n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ G_{n,1}(t) & G_{n,2}(t) & \ldots & G_{n,n}(t) \end{bmatrix}, \tag{3}$$

where $G_{i,j}(t) \in \mathbf{R}^{d\times d}$ for $i, j = 1, \ldots, n$.

We present the main results.

Theorem 1. *System (2) is asymptotically stable if and only if $G_{1,j}(t) \to 0$ as $t \to \infty$, where $G_{1,j}(t)$ is defined in (3) for $j = 1, \ldots, n$.*

Theorem 2. *System (2) is asymptotically stable if and only if $\int_0^\infty \sum_{j=1}^n ||G_{1,j}(t)||_F{}^2 dt < \infty$, where $||\cdot||_F$ stands for the Frobenius norm.*

Theorem 3. *If system (2) is asymptotically stable, then $\int_0^\infty \sum_{j=1}^n ||G_{1,j}(t)||_F{}^2 dt = \frac{1}{2\pi} \int_{-\infty}^\infty \sum_{j=1}^n ||\hat{G}_{1,j}(i\omega)||_F{}^2 d\omega < \infty$, where $\hat{G}_{1,j}(i\omega)$ is the Fourier transform of $G_{1,j}(t)$, for $j = 1, \ldots, n$.*

Theorem 4. *If system (2) is asymptotically stable, then the Fourier transform of $G_{1,j}(t)$ are given*

$$\begin{cases} \hat{G}_{1,j}(i\omega) = (P(i\omega))^{-1}[(i\omega)^{n-j} I + \sum_{l=1}^{n-j} A_l (i\omega)^{n-j-l}] \\ \qquad\qquad\qquad\qquad \textit{for} \quad j = 1, \ldots, n-1 \\ \hat{G}_{1,j}(i\omega) = (P(i\omega))^{-1} \quad \textit{for} \quad j = n. \end{cases}$$

Using **Theorems 3 and 4**, we present an algorithm to design a stabilizing controller of system (1) along the line of [2]. Numerical examples show that the presented algorithm is efficient.

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A NUMERICAL METHOD FOR $\mathcal{H}_2$ MODEL REDUCTION OF LINEAR DELAY SYSTEMS

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Engineering systems such as traffic systems, drilling systems and electric circuits, as well as phenomenon in biology and economics, can often be described by models in terms of delay differential equations. For complex engineering systems, such models might be of high order, i.e., described in terms of a high number of state variables. Due to limited computational, accuracy and storage capabilities, simulation of the full model is often not feasible and necessitating simplification of it. To address the issues of model complexity, this paper presents a numerical method for model reduction of linear delay systems. Consider the following linear time-invariant system with multiple delays with the transfer function G:

$$G: \ \dot{x}(t) = A_0 x(t) + \sum_{d=1}^{m} A_d x(t-\tau_d) + Bu(t), \ y(t) = Cx(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$. We assume that system (1) is asymptotically stable for zero input. The problem we consider is to find a stable reduced-order system that closely approximates the input-output behaviour of system (1). The reduced system will also be a linear delay system of the form

$$\hat{G}: \ \dot{\hat{x}}(t) = \hat{A}_0 \hat{x}(t) + \sum_{d=1}^{m} \hat{A}_d \hat{x}(t-\tau_d) + \hat{B}u(t), \ \hat{y}(t) = \hat{C}\hat{x}(t), \tag{2}$$

where $\hat{x} \in \mathbb{R}^r$ is the state vector of the reduced-order system, $r \ll n$. Our purpose is to determine the matrices $\hat{A}_0, \dots, \hat{A}_m$, $\hat{B}$ and $\hat{C}$ such that the reduced-order system (2) is asymptotically stable in absence of input and the $\mathcal{H}_2$ norm $\|G - \hat{G}\|_{\mathcal{H}_2}$ is as small as possible.

Let $\mathrm{vec}(A)$ represents the column vectorization of a matrix A. Let $\vec{p} = [\mathrm{vec}(\hat{A}_0)^T, \dots, \mathrm{vec}(\hat{A}_m)^T, \mathrm{vec}(\hat{B})^T, \mathrm{vec}(\hat{C})^T]^T$ be the decision variable that contains all the parameters to be determined. Let $G_e(\vec{p}, s) = G(s) - \hat{G}(\vec{p}, s)$. Consider $u(t) = 0$, the state transition matrix of system (2) is denoted by

$X(\vec{p}, t)$, which satisfies

$$\begin{cases} \dot{X}(\vec{p}, t) = \hat{A}_0 X(\vec{p}, t) + \sum_{d=1}^{m} \hat{A}_d X(\vec{p}, t - \tau_d), \ t > 0, \\ X(\vec{p}, t) = 0, \ t < 0 \text{ and } X(\vec{p}, 0) = I. \end{cases} \tag{3}$$

Theorem. *If there exists $\hat{A}_0, \ldots, \hat{A}_m$, $\hat{B}$ and $\hat{C}$ such that*

$$J(\vec{p}) = \lim_{T \to \infty} \int_0^T \alpha \|X(\vec{p}, \sigma)\|_F^2 d\sigma + \lim_{\omega_b \to \infty} \int_{-\omega_b}^{\omega_b} \beta Tr \left[G_e(\vec{p}, j\omega)^* G_e(\vec{p}, j\omega) \right] d\omega$$

exists and is finite, where the weights α and β are positive constants. Then the reduced-order system (2) is asymptotically stable and there exists a bounded positive constant γ such that the $\mathcal{H}_2$ norm $\|G_e\|_{\mathcal{H}_2} = \gamma$.

According to the Theorem, we formulate the following optimization problem to solve $\vec{p}$:

$$\min_{\vec{p}} \alpha \int_0^T \|X(\vec{p}, \sigma)\|_F^2 d\sigma + \beta \int_{-\omega_b}^{\omega_b} \mathrm{Tr} \left[G_e(\vec{p}, i\omega)^* G_e(\vec{p}, i\omega) \right] d\omega \tag{4}$$

subject to the constraints, Eqs. (3),

where $\alpha > 0$ and $\beta > 0$ are weight coefficients, T and ω_b are sufficiently large constants.

By applying numerical discretization methods for the objective function (4) and the constraints (3), the above optimization problem can be reduced to a numerically solvable version. This is a nonlinear programming with equality constraints and can be converted to an unconstraint optimization problem. Various gradient-based algorithms, such as the BFGS quasi-Newton method, can be used for solving this problem.

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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO ONE LINEAR DELAY DIFFERENTIAL EQUATION

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We consider the following delay differential equation:

$$\begin{cases} \dfrac{d}{dt}y(t) = ay(t) + by(t-\tau), \quad t > 0, \\ y(t) = \varphi(t), \quad t \in [-\tau, 0], \\ y(0+) = \varphi(0), \end{cases} \tag{1}$$

where $\varphi \in C([-\tau, 0])$, a, b, $\tau \in \mathbb{R}$ are constants, $\tau > 0$. The aim of this work is to find such a and b that any solution $y(t)$ to problem (1) tends to zero as $t \to \infty$, as well as to obtain an estimate of the rate of this convergence. The following results are obtained.

Theorem. *1) In the required domain, all solutions converge to zero at an exponential rate;*

2) for $b > \max\{0, -a\}$, there exist solutions $y(t)$ to problem (1) infinitely increasing with the growth of t at an exponential rate;

3) for $|b| = -a \geq 0$, any solution $y(t)$ is bounded;

4) for $0 \leq |b| < -a$, any solution $y(t)$ decreases to zero, moreover

$$|y(t)| \leq C \left[\frac{t}{\tau}\right] e^{(-|a| + W(|b|e^{|a|}))\left[\frac{t}{\tau}\right]},$$

where $C > 0$ is a constant that depends on a, b and φ, $W(z)$ is the main branch of the W-Lambert function.

For $b < 0$, a condition on a and b of the convergence to zero of solutions to problem (1) was considered in [1]: $\dfrac{a}{-b} < \cos\left(\sqrt{|b^2 - a^2|}\right)$.

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DIFFUSION-WAVE TYPE SOLUTIONS TO A NONLINEAR SECOND-ORDER PARABOLIC EQUATION

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The report is devoted to constructing and studying a particular class of solutions to second-order nonlinear parabolic equations. We consider the following equation

$$T_t = (\Phi_1(T))_{xx} + (\Phi_2(T))_x + \Phi_3(T), \tag{1}$$

where t, x are independent variables: t is time, x is a spatial variable; $T(t, x)$ is an unknown function, and $\Phi_i,\ i = 1, 2, 3$, are the specified functions.

The most known particular case is the porous medium equation [1], which corresponds to the case where Φ_1 is a power function and $\Phi_2 = \Phi_3 \equiv 0$. If Φ_1 and Φ_3 are power functions and $\Phi_2 \equiv 0$, Eq. (1) becomes the generalized porous medium equation [1]; it is also called "the nonlinear heat equation with a source".

If the functions $\Phi_1(T), \Phi_2(T)$ are differentiable, then (1) can be rewritten as

$$T_t = (\Phi_1'(T)T_x)_x + \Phi_2'(T)T_x + \Phi_3(T). \tag{2}$$

In turn, if the function $K(T) = \Phi_1'(T)$ is sufficiently smooth and monotonic, after the substitution $u = K(T)$, Eq. (2) can be reduced to

$$u_t = uu_{xx} + f(u)u_x^2 + g(u)u_x + h(u), \tag{3}$$

where $f(u) = u\phi''(u)/\phi'(u) + 1$, $g(u) = \Phi_2'(\phi(u))$, $h(u) = \Phi_3(\phi(u))/\phi'(u)$, $K(\phi(u)) = u$, i.e., $\phi(u)$ is the inverse function to $K(T)$.

Consider the boundary condition:

$$u(t, x)|_{x=a(t)} = 0. \tag{4}$$

Problem (3), (4) includes only one boundary condition, which makes the term multiplying the higher derivative vanish. Thus, we cannot apply the classical existence and uniqueness theorems in this case and propose a new one.

Theorem. *Let $a(t)$, $f(u)$, $g(u)$ and $h(u)$ be analytical functions of their arguments. Let the following conditions also hold:*

$$f(0) > 0, \quad g(0) + a'(0) \neq 0, \quad h(0) = 0.$$

Then, problem (3), (4) *has two analytical solutions at the point* $(t = 0, x = a(0))$*: they are the trivial* $u \equiv 0$ *and nontrivial* $u > 0$*, where the latter can be written as a characteristic series. There are no other solutions in the class of analytical functions.*

To prove the theorem we follow the technique by A. F. Sidorov [2]. The solution is constructed in the form of a power series, the convergence of which is proved by the majorant method.

Since the theorem do not allow us to investigate the properties of solutions, we have considered ansatzes that reduce solution construction to Cauchy problems for second-order ODEs with singularity.

Let us construct exact solutions to Eq. (3) using the Clarkson–Kruskal direct method [3–5]; i.e., in the form

$$u = \varphi(t)v(z), \quad z = z(x, t),$$

where $\varphi(t), z(x, t)$ are sufficiently smooth functions.

The obtained form of such solutions will be presented.

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THE NUMERICAL SOLUTION OF THE NONLINEAR MIXED HEAT EQUATION WITH THE ITERATION METHOD

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In the V.N.Khankhasaevs paper, which is bound up with the problem of mathematical modeling of the process of switching off the electric arc in the gas flow, various mathematical models bound up with the hyperbolic equation of heat conductivity (obtained by generalization of the Fourier hypothesis) were studied both analytically and numerically [1].

In course of investigations bound up with the transfer processes in the case of high-intensity influence of the gas, the earlier hypotheses presuming the proportionality of the flow density to the vector of the potential gradient, which are based on the known physics laws, lead to an infinite rate of distribution of the perturbations, what contradicts to fundamental laws of nature.

The approximation of a continuous medium, used in the classical laws, means that in the integral conservation laws for this medium one can make a transition to the limit when the volume tends to zero. This passage to the limit allows us to obtain the energy conservation equation in differential form. From a modern physical point of view, this procedure is incorrect, since the medium always consists of molecules and has its own internal discrete structure [2].

To continue the process of this investigation, let us modify the mathematical model, while considering the hyperbolic-parabolic equation

$$k(x,t)u_{tt} + c_v(x,t)\cdot\rho(x,t)u_t = (\lambda(u,x,t)u_x)_x + c(x,t)u + f(x,t) \qquad (1)$$

in the rectangular domain $G = [0,X]\times[T_1,T_2]$, $T_1 < 0$,$T_2 > 0$. Furthermore, $\forall (x,t)\in G$, $k(x,t) = 0$, $t \le 0$; $k(x,t) > 0$, $t > 0$. That is, when $t \le 0$ the equation (1) is parabolic, and when $t > 0$ it is hyperbolic.

Initial boundary value problem: it is necessary to obtain the solution of equation (1) in the domain G such that

$$u(x,t)|_{t=T_1} = u_0(x); \qquad (2)$$

$$t \in [T_1, T_2] : -\lambda(0, t, u)\frac{\partial u(0, t)}{\partial x} + \alpha_1(tc1 - u(0, t)) = 0; \tag{3}$$

$$\lambda(X, t, u)\frac{\partial u(X, t)}{\partial x} + \alpha_2(tc2 - u(X, t)) = 0. \tag{4}$$

The mathematical model (1)–(4) is solved using the method of difference schemes in the Mathcad-15 software package. The conservative scheme and the simple iteration method are used, the result practically coincides with the experimental data [3–5].

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ASYMPTOTIC PROPERTIES OF SOLUTIONS TO ONE CLASS OF SYSTEMS OF DELAY DIFFERENCE EQUATIONS WITH PERIODIC COEFFICIENTS

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We consider the nonlinear system of delay difference equations

$$x_{n+1} = A(n)x_n + B(n)x_{n-\tau(n)} + F(n, x_n, \dots, x_{n-\tau}), \tag{1}$$

$$n = 0, 1, \dots,$$

where $A(n)$, $B(n)$ are sequences of N-periodic $m \times m$ matrices, $\tau(n) \in \mathbb{N}$ is a delay function, $1 \le \tau(n) \le \tau < \infty$, $F(n, u_0, \dots, u_\tau)$ is a continuous vector function satisfying one of the conditions:

$$1)\ \|F(n, u_0, \dots, u_\tau)\| \le q_0\|u_0\| + \dots + q_\tau\|u_\tau\|, \quad q_i > 0,$$

$$2)\ \|F(n, u_0, u_1, \dots, u_\tau)\| \le q\|u_0\|^{1+\omega}, \quad q,\, \omega > 0.$$

Using a special Lyapunov–Krasovskii functional, sufficient conditions for the asymptotic stability of the zero solution to linear systems of the form (1) $\Big(F(n, x_n, \dots, x_{n-\tau}) \equiv 0\Big)$ were established in [1].

In this paper, for nonlinear systems of the form (1) with periodic coefficients in linear terms, we obtain conditions for the asymptotic stability of the zero solution, estimates for attraction sets and estimates characterizing decay rates of solutions to (1) at infinity [2].

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DIFFERENTIAL EQUATIONS WITH ORDINARY AND INVARIANT DERIVATIVES

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The report discusses the concept of differential equations with ordinary and invariant derivatives.

Definition 1. Differential equation with ordinary derivatives is an equation with respect to a function of one variable and its ordinary and invariant derivatives.

Definition 2. Differential equation with partial derivatives is an equation with respect to unknown function of several variables and its partial derivatives.

Differential equations	
Differential equations with ordinary derivatives	Differential equations with partial derivatives
	• Elliptic PDE, • Parabolic PDE, • Hyperbolic PDE

- Ordinary differential equations

$$\frac{dx(t)}{dt} = g(t, x(t))$$

- Integro-differential equations

$$\dot{x}(t) = \int_a^t K(t, s, x(s))ds = q(t), \quad t \geq a$$

- Delay differential equations

$$\dot{x}(t) = f(t, x(t), x(t-\tau)), \quad \tau = const > 0$$

- Neutral FDEs

$$\dot{x}(t) = \hat{f}(t, x(t), x(t-\tau), \dot{x}(t-\tau)), \quad \tau = const > 0$$

($\dot{x}(t-\tau) \doteq \partial x(t-\tau)$ is the invariant derivative [1])

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HYPERBOLIC EQUATIONS WITH DEGENERACY AND INCREASING LOWER TERMS

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The report presents the results:

— on the solvability of initial-boundary value problems for linear hyperbolic equations with degeneracy;

— on the solvability of initial-boundary value problems for quasi-linear hyperbolic equations with increasing dissipation.

The specificity of the problems studied in the first part is that the degeneracy is uncharacteristic. Previously, such cases have not been studied.

The second part of the report presents the results on the global solvability of various boundary value problems, including problems with nonlinear boundary conditions, for hyperbolic equations with nonlinear power-increasing dissipation. Previously, only results on local solvability were known for the studied equations.

Let us clarify that in all cases, the results were obtained on the existence of regular solutions, i.e., solutions having all Sobolev generalized derivatives included in the corresponding equations.

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EIGENVALUES AND EIGENFUNCTIONS OF DIFFERENTIAL EQUATIONS WITH INVOLUTION

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The direct object of research in this paper is differential equations with an involutive deviation of the argument.

Defined on the segment $[0,T]$ continuously differentiable a monotonically decreasing function $\varphi(t)$ is called an involution if $\varphi(\varphi(t)) = t$ is executed for $t \in [0,T]$. The simplest example of an involution is the fractional linear function

$$\varphi(t) = \frac{a(T-t)}{ct+a}, \quad a, c, T \in \mathbb{R}$$

(in the case of $c = 0$, linear function), when for the numbers a, c and T, the inequality holds $a(cT + a) > 0$; other examples can be found in [1].

The paper shows that the presence of terms with an involutive deviation in the differential equation significantly changes the properties of solutions. In particular, it is shown that the presence in an ordinary first-order differential equation with constant coefficients of a summand with an involutive deviation leads to the appearance of proper functions for the Cauchy problem. Similar results are obtained for some non-local problems, as well as for boundary value problems for parabolic and pseudoparabolic equations with involution.

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SOME CONJUGATION PROBLEMS FOR DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS ALTERNATING COEFFICIENT

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The solvability of the following boundary value problem is investigated for the third-order differential equations with a discontinuous alternating coefficient at the highest derivative of the time variable.

For a bounded domain Ω from the space $\mathbb{R}^n$ with a smooth (for simplicity, infinitely differentiable) boundary Γ, a positive number T is given, Q_1 and Q_2 are cylinders $\Omega\times(-T,0)$ and $\Omega\times(0,T)$ respectively, $\varphi(t)$, $f(x,t)$ are given functions defined at $t\in[-T,T]$, $x\in\overline{\Omega}$, $\alpha=(\alpha_i)$, $\beta=(\beta_i)$, $i=\overline{1,6}$, are given vectors with real coordinates, Δ is Laplace operator on spatial variables, L is differential operator whose action on a given function $v(x,t)$ is defined by the equality

$$Lv=\varphi(t)D_t^3v+\Delta v.$$

Function $u(x,t)$ is a solution to equation

$$Lu=f(x,t)$$

in cylinders Q_1 and Q_2.

In addition, the conjugation conditions are met for function $u(x,t)$:

$$\alpha_1u(x,-0)+\alpha_2u(x,+0)+\alpha_3u_t(x,-0)+\alpha_4u_t(x,+0)$$

$$+\alpha_5u_{tt}(x,-0)+\alpha_6u_{tt}(x,+0)=0,\quad x\in\Omega,$$

$$\beta_1u(x,-0)+\beta_2u(x,+0)+\beta_3u_t(x,-0)+\beta_4u_t(x,+0)$$

$$+\beta_5u_{tt}(x,-0)+\beta_6u_{tt}(x,+0)=0,\quad x\in\Omega,$$

$$u(x,t)|_{\Gamma\times(-T,0)} = 0, \quad u(x,t)|_{\Gamma\times(0,T)} = 0.$$

The existence and uniqueness theorems of regular solutions are proved.

The results of the study of the influence of parameters on the correctness of a certain conjugation problem for a differential equation of the Boussinesq–Love type are also presented. Theorems showing the influence of parameters on uniqueness and non-uniqueness, the existence and non-existence of regular solutions to this problem are proved.

These studies are continuation of works [1–2].

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NUMERICAL SOLUTION OF THE SPIN WAVE TRANSPORT EQUATION USING COMPACT DIFFERENCE SCHEME

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Currently, the field of magnonics is attracting more interest due to the fact that the magnon current is also a means of information transmission. An important direction in magnetic materials is the study of the dynamics of spin wave propagation. Since a spin wave propagating in a magnetic material is damped, it is necessary to enhance the amplitude of the spin wave. Recently the spin-wave amplification model is studied qualitatively by phase–plane methods for linear and nonlinear cases in [1]. The nonlinear transport equation for the magnetostatic spin wave envelope was analyzed by the characteristic method, and the dependence of the amplitude on the nonlinearity coefficient on the phase portraits was established in [2].

Due to the lack of an analytical solution, a quantitative study of the dynamics of the evolution of the initial profile of the spin wave is of interest. For this purpose, the present work considers the application of a fourth-order compact scheme [3] for the numerical solution of the initial model.

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DYNAMICS AND EXACT SOLUTIONS OF EVOLUTIONARY PARTIAL DIFFERENTIAL SYSTEMS

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By systems of evolutionary differential equations we mean systems of the form

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{f}\left(\mathbf{x}, \mathbf{u}, \frac{\partial \mathbf{u}}{\partial \mathbf{x}}, \dots, \frac{\partial^k \mathbf{u}}{\partial \mathbf{x}^k}\right).$$

Here $\mathbf{x} = (x_1, \dots, x_n)$ is a vector of independent spatial variables, t is time, $\mathbf{u} = (u^1, \dots, u^m)$ and $\mathbf{f} = (f^1, \dots, f^m)$ are vector functions. We suppose that the functions $f_1, \dots, f_m$ belongs to the class C^∞ within its domain. The symbol $\partial^i \mathbf{u}/\partial \mathbf{x}^i$ $(i = 1, \dots, k)$ means the set of all partial derivatives of order i by $\mathbf{x}$.

The main idea is as follows.

This system generates a flow on maximal integral manifolds of some completely integrable distributions P [1, 2], i.e., its right parts defines Lie algebra of symmetries of P. Consider the case when the distribution is generated by some overdetermined system of partial differential equations

$$\frac{\partial^{q+1} \mathbf{v}}{\partial \mathbf{x}^{\sigma+1_i}} = \mathbf{V}_{\sigma+1_i}\left(\mathbf{x}, \mathbf{v}, \frac{\partial \mathbf{v}}{\partial \mathbf{x}}, \dots, \frac{\partial^q \mathbf{v}}{\partial \mathbf{x}^q}\right), \ |\sigma| = \sigma_1 + \dots + \sigma_n = q; \ i = 1, \dots n,$$

where $\sigma = (\sigma_1, \dots, \sigma_n)$ is a multi-index, $\sigma_i \in \{0, 1, \dots, q\}$, $|\sigma| = \sigma_1 + \dots + \sigma_n$, $\sigma + 1_i = (\sigma_1, \dots, \sigma_{i-1}, \sigma_i + 1, \sigma_{i+1}, \dots, \sigma_n)$ $\mathbf{v}$ is a vector-valued function of $\mathbf{x} = (x_1, \dots, x_n)$.

Let S be a shuffling symmetry of the distribution P [3]. There are a unique set of functions $\varphi^1, \dots, \varphi^m$ on J^q such that

$$S = \sum_{j=1}^{m} \varphi^j \frac{\partial}{\partial v_o^j} + \sum_{\substack{|\sigma|=1 \\ j=1,\dots,m}} \mathcal{D}^\sigma(\varphi^j) \frac{\partial}{\partial v_\sigma^j} + \dots + \sum_{\substack{|\sigma|=q \\ j=1,\dots,m}} \mathcal{D}^\sigma(\varphi^j) \frac{\partial}{\partial v_\sigma^j}.$$

Here $o = (0, \ldots, 0)$ is zero multi-index, $\mathcal{D}^\sigma = \mathcal{D}_1^{\sigma_1} \circ \cdots \circ \mathcal{D}_n^{\sigma_n}$, and $\mathcal{D}_i^s$ is the s-th degree of the operator

$$\mathcal{D}_i = \frac{\partial}{\partial x_i} + \sum_{\substack{0 \le |\sigma| \le q \\ j=1,\ldots,m}} v^j_{\sigma+1_i} \frac{\partial}{\partial v^j_\sigma} + \sum_{\substack{0 \le |\sigma| = q \\ j=1,\ldots,m}} V^j_{\sigma+1_i}(\mathbf{x}, \mathbf{v}_\sigma) \frac{\partial}{\partial v^j_\sigma} \quad (i = 1, \ldots n).$$

Note that the distribution P is generated by the vector fields $\mathcal{D}_1, \ldots, \mathcal{D}_n$.

The functions $\varphi^1, \ldots, \varphi^m$ satisfy the following system:

$$\mathcal{D}^{\sigma+1_i}(\varphi^j) - \sum_{s=1}^{n} \sum_{|\mu|=0}^{q} \mathcal{D}^\mu(\varphi^s) \frac{\partial V^j_{\sigma+1_i}}{\partial v^s_\mu} = 0, \quad i = 1, \ldots, n; \quad j = 1, \ldots, m.$$

Solving this system we can find the vector field S. Shifts along this vector field of solutions of the overdetermined system, we obtain a solution to the evolutionary system.

This method will be illustrated using the examples of the Boussinesq equation [4]

$$\begin{cases} u_t = u_{xx} + 2v_x, \\ v_t = -v_{xx} + 2uu_x - 2u_y. \end{cases}$$

It made it possible to construct a family of exact solutions of the Boussinesq equation which depends on six arbitrary parameters and one arbitrary function.

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MATHEMATICAL MODELING OF THE DYNAMICS OF CO OXIDATION OVER Pt/MWCNB

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To describe the dynamics of chemical reactions, one or another reaction mechanism is assumed, which is the basis for constructing a kinetic model. Modern physico-chemical methods of surface research indicate the need to take into account various structures or active phases in the model. Thus, there is a need to construct and study systems of nonlinear ordinary differential equations.

In particular, in our work [1], a mathematical model was proposed describing the temperature dependencies of the concentration of CO in the gas phase under the conditions of the CO oxidation on platinum supported on multilayer carbon nanotubes (MWCNB) in a continuous stirred-tank reactor:

$$\begin{aligned}
&\tfrac{d\theta_{CO}}{dt} = k_1 P_{CO}\theta_f - k_{-1}\theta_{CO} - k_3\theta_{O2}\theta_{CO},\\
&\tfrac{d\theta_{O2}}{dt} = k_2\theta_f - k_3\theta_{O2}\theta_{CO} - k_4\theta_{O2}\theta_f,\\
&\tfrac{d\theta_{O}}{dt} = 2k_3\theta_{O2}\theta_{CO} - k_5\theta_O,\\
&\tfrac{d\theta_{Ox}}{dt} = k_5\theta_O - k_6 P_{CO}\theta_{Ox},\\
&\tfrac{dP_{CO}}{dt} = \alpha(P^0_{CO} - P_{CO}) - \gamma(k_1 P_{CO}\theta_f - k_{-1}\theta_{CO} + k_6 P_{CO}\theta_{Ox}),
\end{aligned} \tag{1}$$

where $\theta_f = 1 - \theta_{CO} - \theta_{O2} - \theta_O - \theta_{Ox} \geq 0$ and the model variables are non-negative. The parameters of the model $k_i, \alpha, \gamma, P^0_{CO} \geq 0$.

As a result of parametric analysis, parameter regions were identified in which the system has a single steady-state. In addition, as a result of numerical analysis, it is shown that for some parameters values, system (1) has a stable limit cycle.

Moreover, system (1) was studied under the assumption that part of the parameters k_i of the system depends on the temperature T according to the Arrhenius law, $k_i = k_{i,0} exp(-E_i/(RT))$, where R is the universal gas constant.

As a result of the application of numerical methods, it is shown that for some parameter values, the system describes the so-called dynamic hysteresis under conditions of heating and subsequent cooling of the catalyst.

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OPTIMAL CONTROL OF THE ANGLE BETWEEN TWO THIN RIGID INCLUSIONS IN AN INHOMOGENEOUS 2D BODY

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Mathematical models of elastic bodies with cracks subject to unilateral boundary conditions of Signorini's type on curves or surfaces of cracks, have been actively studied since the 1990s (see, for example, [1–3]). A nonlinear mathematical model describing equilibrium of a two-dimensional elastic body with two thin rigid inclusions is investigated. It is assumed that two rigid inclusions have one common connection point. Moreover, a connection between two inclusions at a given point is characterized by a positive damage parameter. Rectilinear inclusions are located at a given angle to each other in an initial state. A nonlinear Signorini condition is imposed, which describes the contact with the obstacle, as well as a homogeneous Dirichlet condition is set on corresponding parts of the outer boundary of the body. An optimal control problem for the parameter that specifies the angle between inclusions is formulated. The quality functional is given by an arbitrary continuous functional defined on the Sobolev space. The solvability of the optimal control problem is proved. A continuous dependence of solutions on varying angle parameter between the inclusions is established.

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A SPACE-TIME SPECTRAL METHOD FOR A ONE-DIMENSIONAL PARABOLIC INVERSE PROBLEM

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In this work, a space-time spectral method based on a Legendre–Galerkin–Chebyshev collocation in space and a Legendre-tau method in time is presented for a parabolic inverse problem with control parameters. The nonlinear term is collocated at the Chebyshev–Gauss–Lobatto points implemented by the fast Legendre transform. Suitable basis functions are used in each direction, which yields an algebraic system of sparse matrices. The approximation results of the Chebyshev interpolation operator in the Legendre norm are given. Numerical examples are compared with some other methods to confirm the efficiency and capability of our method.

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HOMOGENIZATION MODELING OF GENERALIZED NEWTONIAN FLUID FLOW IN POROUS MEDIA

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The non-Newtonian flow in porous medium has attracted much attention due to its important role in composite materials and petroleum industry. However, due to the spatial multi-scale of porous medium and the rheological properties of fluids, this flow mechanism is very complex.

This report mainly introduces the mathematical modeling of generalized Newtonian fluid flow in porous medium using asymptotic homogenization method. The local problem on periodic cells is obtained to describe the local transmission of generalized Newtonian fluid in pores. Through theoretical analysis of local problems, the permeability tensor of Generalized Newtonian fluid is obtained, which is proved to be symmetric and positive definite. A least squares finite element numerical solution for local problems has been developed based on the physical properties of microscopic pore structures. The solution of local problems can not only determine the accurate distribution of velocity, pressure and non Newtonian viscosity in a single hole, but also evaluate the permeability coefficient and effective viscosity of generalized Newtonian fluid in porous medium.

The micro flow of Carreau–Yasuda fluid in three-dimensional porous ceramics was simulated, and the proposed mathematical model and numerical method were validated. The sensitivity of non Newtonian viscosity to permeability and effective viscosity was discussed through numerical simulation.

Theorem. *The solution $V_i^{(j)}$ of the local problem has the following relationship with the local problem:*

$$\langle v_i^{(0)} \rangle = K_i^j \frac{g_j - p_{,x_j}^{(0)}}{|g_j - p_{,x_j}^{(0)}|},$$

where $K_i^j = \int_{\Omega_{yf}} V_i^{(j)} d\mathbf{y}$ is the permeability tensor and K_i^j is symmetric positive definite. Since $V_i^{(j)}(\mathbf{y}, |g_j - p_{,x_j}^{(0)}|)$ are nonlinear vector functions, *in fact the tensor* $K_i^j(|g_j - p_{,x_j}^{(0)}|)$ *is a symmetric positive definite tensor composed of nonlinear tensor functions.*

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EXACT SOLUTION OF THE PROBLEM ON FORCED VIBRATIONS OF A STRING OF VARIABLE LENGTH

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The article considers the resonant characteristics of forced vibrations of a string with moving boundaries. The analytical method was developed in relation to obtaining an exact solution of the wave equation with a higher class of conditions on moving boundaries that differ from the boundary conditions of the first kind.

The differential equation describing forced strings is:

$$Z_{tt}(x,t) - a^2 Z_{xx}(x,t) = \omega_0^2 B \cos W_0\big(\omega_0 t\big).$$

Border conditions:

$$Z(0,t) = 0; \quad Z\big(l_0(t),t\big) = 0.$$

The initial conditions do not affect the resonant properties of linear systems, so they are not considered in this problem [1]. Performing transformations, similar to transformations [2, 3], we obtain an expression for the total amplitude at the point $\xi = \xi_0(\tau)$, corresponding to the maximum amplitude of oscillations

$$A_n^2(\tau) = \left\{ \left[\int\limits_0^{b(\tau)} F_n(\zeta) \cos \Phi_n(\zeta) d\zeta \right]^2 + \left[\int\limits_0^{b(\tau)} F_n(\zeta) \sin \Phi_n(\zeta) d\zeta \right]^2 \right\}.$$

Thus, an expression for the amplitude of system oscillations in the nth dynamic mode has been obtained using the application of special functional equations.

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ON THE VARIATIONAL PRONY PROBLEM WHEN DATA CONTAINS DISTURBANCES LYING ON A GIVEN LINEAR MANIFOLD

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We study the variational Prony problem of a linear difference equation coefficients identification when data contains disturbances lying on a given linear manifold. The projectivity and the consistency properties of the target function are proven, and the numeric algorithm for finding global minimum based on inverse iterations is proposed. Formulas for optimal filtering of disturbances and additive noise are given. Numeric results are presented.

The system under discussion has the form (for $k = \overline{1, N-n}$)

$$x_{k+n} + \alpha_{n-1} x_{k+n-1} + \dots + \alpha_0 x_k = \beta_n u_{k+n} + \dots + \beta_0 u_k, \qquad (1)$$

or: $\quad Gz = 0, \quad z \doteq [x_1; u_1; \ldots; x_N; u_N], \quad G = \operatorname{toeplitz}_{(N-n)\times 2N}[\alpha_i, -\beta_i].$

The measured data $\check{z} = z + \eta + s \in \mathbb{R}^{2N}$ contains random η and deterministic $s = S\pi$ terms with given matrix $S : \forall\theta \ \operatorname{im} S \cap \ker G_\theta = 0$, $\theta \doteq [\alpha; \beta] \in \mathbb{R}^{2n+1}$.

The variational Prony problem [1–3] is to estimate z, π [4] and θ minimizing target function [3]

$$J = (\check{z} - z - S\pi)^\top (\check{z} - z - S\pi) \to \min_{\pi, z, \theta :\, G_\theta z = 0}.$$

Theorem 1. *Let $\mathit{\Pi}_\theta \doteq G^\top \left(GG^\top\right)^{-1} G$. The minimum point of J is*

$$\hat{\theta} = \arg\min_\theta \check{z}^\top \left(\mathit{\Pi}_\theta - \tilde{\mathit{\Pi}}_\theta\right) \check{z}, \quad \tilde{\mathit{\Pi}}_\theta \doteq \tilde{S}\left(\tilde{S}^\top \tilde{S}\right)^{-1} \tilde{S}^\top, \quad \tilde{S} \doteq \mathit{\Pi}_\theta S,$$

$$\hat{\pi} = \left(\tilde{S}^\top \tilde{S}\right)^{-1} \tilde{S}^\top \mathit{\Pi}_{\hat{\theta}} \check{z}, \quad \hat{z} = \left(I - \mathit{\Pi}_{\hat{\theta}}\right)(\check{z} - S\hat{\pi}).$$

A finite algorithm for calculating the approximate value of θ lying in a small $\hat{\theta}$-neighborhood $E \subset \mathbb{R}^{2n+1}$ ($\operatorname{diam} E \underset{\|\eta\|\to 0}{\longrightarrow} 0$) is proposed.

Uniqueness conditions are obtained. Let H_θ be the basis of $\ker G_\theta$.

Definition 1. Let's call the vector $w \doteq [w_1; \ldots; w_{N-n}]$ the impulse response function of the system (1) if $w = [x_{n+1}; \ldots; x_N]$, where $[x_{n+1}; \ldots; x_N]$ is the solution of the system (1) with the function $u = [1; 0; \ldots; 0]$ in the right side under zero initial conditions $[x_1; \ldots; x_n] = [0; \ldots; 0]$.

Theorem 2. *In system* (1), *the θ parameter is locally identifiable if any of the three sufficient conditions for the domain $\theta \in \Theta$ is met: 1) the vector α is fixed; 2) the vector β is fixed; 3) the allowable increments* $\mathrm{d}\alpha$, $\mathrm{d}\beta$ *are not connected by the linear relation* $\mathrm{d}\beta = \begin{bmatrix} w_1 & \cdots & w_{n+1} \\ & \ddots & \vdots \\ 0 & & w_1 \end{bmatrix} \mathrm{d}\alpha$.

Definition 2. A vector (grid function) $x \in \mathbb{R}^N$ will be called a quasi-polynomial of degree $n(< N-1)$ if it is the solution of some homogeneous difference equation (1) of order n: $\exists\theta : G_\theta x = 0$, and at the same time there is no equation (1) of a smaller order $m < n$, the solution of which would be the vector x.

Denote $Q_n \subset \mathbb{R}^N$ the set of all quasi-polynomials of order $\leqslant n$.

Theorem 3. *If the columns s_i of the matrix $S \doteq [s_1, \ldots, s_q] \subset \mathbb{R}^N$ are not quasi-polynomials of degree $2n$ or lower, i. e. $s_i \notin Q_{2n}, i = \overline{1, q}$, then in a homogeneous system* (1) *the parameter $\theta \doteq [\alpha_0; \ldots; \alpha_{n-1}]$ is globally identifiable.*

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TAYLOR DECOMPOSITION AND DISCRETE ANALYTIC FUNCTIONS OF PARABOLIC TYPE

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The aim of this paper is to establish the existence and singularity theorems and singularity for a discrete analytic function of parabolic type in the positive quadrant of the Gaussian plane. Let $\mathbb{G}^+ = \{\mathbb{Z}^+ + i\,\mathbb{Z}^+\}$. Denote $\mathcal{A}(\mathbb{C})$ and $\mathcal{D}(\mathbb{G}^+)$ the spaces of analytic functions of exponential type and the discrete analytic functions of parabolic type, defined in $\mathbb{C}$ and $\mathbb{G}^+$ respectively. For the exponent $e(\zeta, z) = e(\zeta, x, y) = e^{\zeta x}\big((e^{\zeta}-1)^2+1\big)^y$, define the pseudo-degrees $\{\pi_k(z)\}_{k=0}^{\infty}$, by the formula $e(\zeta, x, y) = \sum\limits_{k=0}^{\infty} \frac{\pi_k(z)}{k!}\,\zeta^k$ for $\zeta \in \mathbb{C}$ and $z = x + i\,y \in \mathbb{G}^+$.

Let us consider the mapping

$$\begin{gathered}\Theta\colon F(\zeta) \to f(z),\\ \Theta\Big(\sum_{k=0}^{\infty} a_k\zeta^k\Big) = \sum_{k=0}^{\infty} a_k\pi_k(z).\end{gathered} \tag{1}$$

The following theorem is proved in [1].

Theorem. *The mapping* Θ*:* $\mathcal{A}(\mathbb{C}) \to \mathcal{D}(\mathbb{G}^+)$*, defined by formula* (1) *is surjective. The core* $\operatorname{Ker}\Theta$ *of this mapping consists of integer functions* $F(\zeta)$*, which is of the form:*

$$F(\zeta) = \frac{H(\zeta)}{\Gamma(-\zeta)},$$

where $H(\zeta)$ *is an arbitrary integer function and* $\Gamma(\zeta) = \int\limits_0^{+\infty} t^{\zeta-1}e^{-t}dt$ *is the Euler gamma-function.*

In paper [2] the similar theorem was proved for discrete analytic functions of the second type.

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REPRESENTATION OF SOLUTIONS OF THE NONLINEAR EQUATIONS IN THE RING RESONATOR

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The mathematical model in the ring resonator with coordinate transformation, which consists of nonlinear partial differential equations [1]

$$u_t + u = D\triangle u + K|A(r, z = 0, t)|^2, \;\; r = (x, y), \tag{1}$$

$$A(r, z = 0, t + t_r) = (1 - R)^{\frac{1}{2}} A_{in}(r) + Re^{i\varphi_0} \exp iL\triangle\{A(r, z = 0, t)e^{iu(r,t)}\}, \tag{2}$$

$$-2ik_0 \frac{\partial A(r, z, t)}{dz} = \triangle A(r, z, t), \;\; A(r, z = 0, t) = A_0(r, t), \tag{3}$$

is considered in the paper. Here $u(r, t)$ is an unknown phase modulation function, which describes the phase shift of a light wave in a nonlinear medium; $r = (x, y)$ is the radius-vector in the cross-section of the light field; z is the longitudinal coordinate; t is the time; $\triangle$ is the Laplacian, which describes the diffusion process in a nonlinear medium; D is the normalized diffusion coefficient; K is the coefficient of nonlinearity of the medium; $|A(r, z = 0, t)|^2$ is the intensity of the light field which is incident on the nonlinear medium; $A(r, z, t)$ is unknown function, which describes the complex slowly varying amplitude of the light field inside the resonator; R is the reflection coefficient of the mirror intensity; $A_{in}(r)$ is the complex amplitude of the input light wave; t_r is the time of field propagation in the resonator; φ_0 is the constant phase shift of the light wave in the resonator; L is the resonator length; $\exp(iL\triangle)$ is the spread operator.

For (1)–(3) stationary solutions u_s and A_s were obtained:

$$u_s = \frac{(1 - R)k}{1 - 2R\cos(u_s + \varphi_0) + R^2}, \;\; k = K\, I_0, \;\; I_0 = |A_{ins}|^2.$$

The corresponding linearized initial-boundary value problems for the general area and for the circle with unknown functions $u = u_s + v$, $A = A_s + B$ were considered [3]. The original model (1)–(3) can be reduced to the nonlinear integral equation [2].

Theorem. *Solutions of the problem* (1)–(3) *in the nonlinear integral form can be represented as:*

$$v(\theta,t) = \sum_{n=-\infty}^{\infty} (v_n e^{\omega(n)t} + f_n)e^{in\theta} = f(\theta) + \sum_{n=-\infty}^{\infty} v_n e^{\omega(n)t+in\theta},$$

$$B(\theta,t) = \sum_{n=-\infty}^{\infty} (\lambda_n v_n e^{\omega(n)t} + \alpha_n f_n)e^{in\theta},$$

$$\alpha_n = \frac{(1-R)^{1/2} + iA_s R(1-i\rho n^2)e^{i(u_s+\varphi_0)}}{1-R(1-i\rho n^2)e^{i(u_s+\varphi_0)}},$$

$$\lambda_n = \frac{iA_s R(1-\rho n^2)e^{i(u_s+\varphi_0)}}{1-R(1-i\rho n^2)e^{i(u_s+\varphi_0)}},$$

$$\omega_c(n) = -1 - \mu^2 n^2 + \frac{4RK|A_s|^2(\sin(u_s+\varphi_0) + \rho n^2\cos(u_s+\varphi_0))}{1-2R(1+\rho n^2)\cos(u_s+\varphi_0)+R^2}$$

or in the integral representation

$$v(\theta,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} k(\theta-\xi,t)v(\xi)d\xi + f(\theta),$$

$$B(\theta,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} k(\theta-\xi,t)v(\xi)d\xi + \frac{1}{2\pi}\int_{-\pi}^{\pi} a(\theta-\xi)f(\xi)d\xi,$$

$$k(\theta,t) = W\{\lambda_n e^{\omega(n)t}\}(\theta,t), \;\; a(\theta) = W\{\alpha_n\}(\theta).$$

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THE STABILITY OF THE SOLUTIONS TO STATIONARY INVERSE PROBLEMS

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The inverse problems on recovering the unknown lower coefficient in linear and nonlinear second-order elliptic equations with boundary integral conditions of overdetermination are considered. Namely, for given functions $f(x), \beta(x), h(x)$ and constant μ we find the function u and constant k satisfying the equation

$$-\operatorname{div}(\mathcal{M}(x)\nabla u) + m(x)u + kr(u) = f, \tag{1}$$

almost everywhere in Ω, the boundary condition

$$B_1 u|_{\partial\Omega} = \beta(x), \tag{2}$$

and the condition of overdetermination

$$\int_{\partial\Omega} B_2 u\, h(x)\, ds = \mu. \tag{3}$$

Here $\Omega \cap \mathbf{R}^n$ is a bounded domain with the boundary $\partial\Omega \in C^2$, $\mathcal{M}(x) = m_{ij}(x)$ is a matrix of functions $m_{ij}(x)$, $i, j = 1, 2, \ldots, n$, $m(x)$ is a scalar function, $r(s)$ is a continuous function defined on $(-\infty, +\infty)$; B_1, B_2 are the linear differential operators acting in the spaces of functions defined on the boundary $\partial\Omega$. The operator $M = -\operatorname{div}(\mathcal{M}(x)\nabla) + m(x)I : W_2^1(\Omega) \to (W_2^1(\Omega))^*$ is supposed to be strongly elliptic (I is the identity operator).

The continuous dependence of the strong solution on the input data of the inverse problem (1)–(3) is proved in three cases: 1) $r(s) = s$, $B_1 u = u$, $B_2 u = \frac{\partial u}{\partial N} = (\mathcal{M}(x)\nabla u, \mathbf{n})$, $\mathbf{n}$ is the unit vector of the outward normal to the boundary $\partial\Omega$; 2) $r(s) = s$, $B_1 u = \frac{\partial u}{\partial N} + \sigma(x)u$, $B_2 u = u$; 3) $r(s)$ is a nonlinear monotone power-type function, $B_1 u = u$, $B_2 u = \frac{\partial u}{\partial N}$.

In the hypotheses of the existence and uniqueness theorems [1, 2] the solution of the problem (1)–(3) continuously depends on the input data f,

β, h and μ in cases 1) and 2). The estimate

$$\|u_1-u_2\|_2+|k_1-k_2| \leq K\big(|\mu_1-\mu_2|+\|f_1-f_2\|+\|\beta_1-\beta_2\|_{j+1/2}+\|h_1-h_2\|_{1/2}\big)$$

holds, where $\{u_i, k_i\}$ is the unique solution of the problem (1)–(3) with $f = f_i$, $\beta = \beta_i$, $h = h_i$ and $\mu = \mu_i$, $i = 1, 2$; K is a positive constant; $j = 1$ for case 1) and $j = 0$ for case 2). In case 3), under the hypotheses of the existence and uniqueness theorem [3] the solution of the problem (1)–(3) continuously depends on μ and the estimate

$$\|u_1 - u_2\|_2 + |k_1 - k_2| \leq H|\mu_1 - \mu_2|$$

holds, where $H > 0$ is a positive constant.

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FINITE TIME STABILIZATION TO ZERO AND EXPONENTIAL STABILITY OF QUASILINEAR HYPERBOLIC SYSTEMS

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In the semistrip $\Pi = \{(x,t) : 0 \le x \le 1, 0 \le t < \infty\}$ we consider the following initial-boundary value problem for nonautonomous first-order quasilinear hyperbolic system

$$\begin{gathered}\partial_t u_j + A_j(x,t,u)\partial_x u_j + \sum_{k=1}^{n} B_{jk}(x,t,u)u_k = 0,\\ 0 < x < 1, \quad t > 0, \quad 1 \le j \le n,\end{gathered} \tag{1}$$

with the reflection boundary conditions

$$\begin{aligned} u_j(0,t) &= \sum_{k=1}^{m} p_{jk}u_k(1,t) + \sum_{k=m+1}^{n} p_{jk}u_k(0,t), \quad t \ge 0, \quad 1 \le j \le m,\\ u_j(1,t) &= \sum_{k=1}^{m} p_{jk}u_k(1,t) + \sum_{k=m+1}^{n} p_{jk}u_k(0,t), \quad t \ge 0, \quad m < j \le n,\end{aligned} \tag{2}$$

and the initial conditions

$$u_j(x,0) = \varphi_j(x), \quad 0 \le x \le 1, j \le n, \tag{3}$$

where $n \ge 2$ and $0 \le m \le n$ are fixed integers. The unknown function $u = (u_1,\dots,u_n)$ and the initial function $\varphi = (\varphi_1,\dots,\varphi_n)$ are vectors of real-valued functions. The coefficients A_j and B_{jk} are real-valued smooth functions and the $n \times n$ matrix $P = (p_{jk})$ has real constants. The functions A_j for all $(x,t) \in \Pi$ and for all $\Theta = (\theta_1,\dots,\theta_n)$ are supposed to satisfy the following conditions:

$$A_j(x,t,\Theta) \ge \Lambda_0,\ 1 \le j \le m, \quad A_j(x,t,\Theta) \le -\Lambda_0,\ m+1 \le j \le n, \tag{4}$$

for some $\Lambda_0 > 0$.

The linear nonautonomous problem (1)–(3), namely $A_j(x,t,U) \equiv A_j(x,t)$, $B_{jk}(x,t,U) \equiv B_{jk}(x,t)$, is investigated in [1, 2]. Let A_j and B_{jk} belong to

$C^1(\Pi)$ and be bounded in Π together with their first order derivatives. From theorem 1.8 in [1] we have this proposition.

Theorem 1. *Let the linear problem* (1)–(3) *be decoupled, namely* $B_{jk} = 0$ *for* $j \neq k$*. Then there exists a positive real* T_e *such that for any* A_j *satisfying* (4) *and for any* B_{jj} *all classical solutions to this problem are constant zero functions for all* $t > T_e$ *if and only if the matrix* P_{abs} *is nilpotent, that is*

$$(P_{abs})^n = 0, \quad \text{where } P_{abs} = (|p_{ij}|)_{i,j=1,\dots,n}. \tag{5}$$

From theorem 2.7 in [2] we have this proposition.

Theorem 2. *Let the linear problem* (1)–(3) *be strictly hyperbolic, namely*

$$A_1(x,t) > \dots > A_m(x,t) > 0 > A_{m+1}(x,t) > \dots > A_n(x,t),$$

and the coefficients in the system (1), (2) *fulfill the conditions* (4), (5), *then for any* $\gamma > 0$ *there exist* $\epsilon > 0$ *and* $M = M(\gamma) \geq 1$ *such that, whenever* $\max_{j,k}(\sup_{x,t\in\Pi}(|B_{j,k}(x,t)|, |\partial_t B_{j,k}(x,t)|, |\partial_x B_{j,k}(x,t)|)) < \epsilon$*, the classical solution* u *to problem* (1)–(3) *fullfils the bound*

$$\|u(\cdot,t)\|_{C^1[0,1]} \leq M e^{-\gamma t} \|u_0\|_{C^1[0,1]}, \qquad t \geq 0.$$

The paper [3] deals with asymptotic properties of solutions to initial-boundary value problems (1)–(3) for nonautonomous first-order quasilinear hyperbolic systems with two variables. Case of smoothing boundary conditions (5) is considered. For decoupled hyperbolic systems we prove that all smooth solutions stabilize to zero for finite time not depending on the initial function. For non-decoupled strictly hyperbolic systems we prove that zero solution to quasilinear problem is exponentially stable.

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WEAK SOLUTIONS TO AN INITIAL-BOUNDARY VALUE PROBLEM OF A CONSISTENT THREE-PHASE-FIELD MODEL

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In this topic, we shall investigate the global existence, uniqueness and regularity of weak solutions to an initial-boundary value problem for a three-phase-field model which is proposed to simulate the evolution of Greenland ice-sheet based on criteria that lead to both physical and mathematical consistency. For global solutions, we calculate the global-in-time weak solutions by applying the method of continuation of local solutions.

We establish the consistent three-phase-field model which contains of four second order non-conserved parabolic partial differential equations with temperature based on multi-phase-field theories [1] and the free energy functional firstly established by Steinbach I. et al [2] for multi-phase systems in 1996.

The consistent three-phase-field problem is as follows

$$\begin{cases} \frac{\partial u_1}{\partial t} - (\alpha_1 + \alpha_2)\Delta u_1 = -\alpha_2 \Delta u_2 - \alpha_1 \Delta u_3 + \beta_1 f_1(u_1, u_2, u_3, u_4) \\ \qquad\qquad -\beta_2 f_2(u_1, u_2, u_3, u_4), \ \ (t, \mathbf{x}) \in (0, T) \times \Omega, \\ \frac{\partial u_2}{\partial t} - (\alpha_2 + \alpha_3)\Delta u_2 = -\alpha_2 \Delta u_1 - \alpha_3 \Delta u_3 + \beta_2 f_2(u_1, u_2, u_3, u_4) \\ \qquad\qquad -\beta_3 f_3(u_1, u_2, u_3, u_4), \ \ (t, \mathbf{x}) \in (0, T) \times \Omega, \\ \frac{\partial u_3}{\partial t} - (\alpha_1 + \alpha_3)\Delta u_3 = -\alpha_1 \Delta u_1 - \alpha_3 \Delta u_2 - \beta_1 f_1(u_1, u_2, u_3, u_4) \\ \qquad\qquad +\beta_3 f_3(u_1, u_2, u_3, u_4), \ \ (t, \mathbf{x}) \in (0, T) \times \Omega, \\ \frac{\partial u_4}{\partial t} - D\Delta u_4 = r_1 \frac{\partial u_1}{\partial t} + r_2 \frac{\partial u_2}{\partial t} + r_3 \frac{\partial u_3}{\partial t} + f(x, t), \\ \qquad\qquad (t, \mathbf{x}) \in (0, T) \times \Omega, \\ u_i(0, x) = u_{i0}, \ u_{10} + u_{20} + u_{30} = 1, \qquad x \in \Omega, \\ \frac{\partial u_i}{\partial \mathbf{n}} = 0 \text{ for } i = 1, 2, 3, 4, \qquad (t, \mathbf{x}) \in [0, T] \times \partial\Omega, \end{cases} \tag{1}$$

here $T > 0$, $\mathbf{n}$ denotes the outer unitary normal vector of $\partial\Omega$,

$$f_1 = u_3^3 - u_1^3 + u_3 u_1^2 + u_3 u_2^2 - u_3^2 - u_1 u_2^2 - u_1 u_3^2 + u_1^2 + e u_3 u_4 - e u_1 u_4,$$
$$f_2 = u_1^3 - u_2^3 + u_1 u_2^2 + u_1 u_3^2 - u_1^2 - u_2 u_1^2 - u_2 u_3^2 + u_2^2 + e u_1 u_4 - e u_2 u_4,$$
$$f_3 = u_2^3 - u_3^3 + u_2 u_3^2 + u_2 u_1^2 - u_2^2 - u_3 u_1^2 - u_3 u_2^2 + u_3^2 + e u_2 u_4 - e u_3 u_4.$$

Firstly, we give the definition of weak solutions, then by using Banach fixed-point Theorem, we derive the existence of local weak solutions in a closed ball. Soon afterwards, we show uniform a *priori* estimate to obtain the global-in-time solutions. Finally, we investigate the regularity of the global weak solutions under some assumptions.

Theorem 1. *Suppose open bounded region* $\Omega \in \mathbb{R}^3$*, initial values* u_{10}*,* u_{20}*,* u_{30} *and* $u_{40} \in H^1(\Omega)$ *and for any positive time* T*,* $f \in L^\infty(0,T;L^2(\Omega))$*, on the conditions that we choose a suitable* $m > 0$ *such that* $0 < \alpha_1$*,* α_2*,* α_3*,* $|r_1|$*,* $|r_2|$*,* $|r_3| < m$*, there exists a unique global solution* $(u_1(t,\mathbf{x}), u_2(t,\mathbf{x}), u_3(t,\mathbf{x}), u_4(t,\mathbf{x})) \in (L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)))^4$ *and* $\frac{\partial u_i}{\partial t} \in L^2(Q_T)$ $(i = 1, 2, 3, 4)$ *for the initial-boundary value problem* (1).

Theorem 2. *Assume* $\Omega \in \mathbb{R}^3$ *is open bounded, initial value* $(u_{10}, u_{20}, u_{30}, u_{40}) \in (H^2(\Omega))^4$*,* $(\frac{\partial u_{10}}{\partial t}, \frac{\partial u_{20}}{\partial t}, \frac{\partial u_{30}}{\partial t}, \frac{\partial u_{40}}{\partial t}) \in (L^2(\Omega))^4$ *and for any* $T > 0$*,* $f \in L^\infty(0,T;L^2(\Omega))$*,* $\frac{\partial f}{\partial t}$*,* $\nabla f \in L^2(0,T;L^2(\Omega))$*, then there exists a unique global solution* $(u_1(t,\mathbf{x}), u_2(t,\mathbf{x}), u_3(t,\mathbf{x}), u_4(t,\mathbf{x})) \in (L^\infty(0,T;H^2(\Omega)) \cap L^2(0,T;H^3(\Omega)))^4$ *and* $\frac{\partial u_i}{\partial t} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ $(i = 1, 2, 3, 4)$ *for the initial-boundary value problem* (1) *under some assumptions.*

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ON BEHAVIOR OF SOLUTIONS TO A SOBOLEV TYPE EQUATION AT INFINITY

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We consider the Cauchy problem for one equation unsolvable with respect to the highest time derivative

$$(\Delta - \alpha I)u_{tt} + \omega^2 u_{x_n x_n} = 0, \quad t > 0, \quad x \in R^n \quad (n = 2, 3), \tag{1}$$

$\alpha > 0$, Δ is the Laplace operator in x. This equation for $\alpha = 0$ gives us the Sobolev equation

$$\Delta u_{tt} + \omega^2 u_{x_n x_n} = 0. \tag{2}$$

It describes small vibrations of a rotating fluid for $n = 3$. Our aim is to study the asymptotic behavior of a solution to the Cauchy problem for (1) as $t \to \infty$.

A systematic study of properties of solutions to equations unsolvable with respect to the highest order derivative was begun in the works of S. L. Sobolev [1].

We continue investigations begun in [2] and prove theorems on asymptotic behavior of solutions to the Cauchy problem for (1) as $t \to \infty$. Namely, the form of a limit function is established and a rate of convergence is obtained.

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ON A CONSTRUCTION OF RIEMANNIAN METRICS IN MODELS OF INHOMOGENEOUS MEDIA

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In many mathematical models developed within the framework of the ray approximation for the purpose of studying direct and inverse problems posed in inhomogeneous media, the behavior of rays is described by geodesics of the Riemannian metric. Metrics are considered known in direct problems, and they are unknown in inverse problems. One of the approaches to solving inverse problems in which geodesics arise is to linearize it. Namely, it is assumed that the behavior of geodesics of an unknown metric differs slightly from the behavior of geodesics of some a priori given Riemannian metric.

In seismic problems, the density of an elastic medium as a whole increases noticeably with depth [1]. If there are good reasons to believe that the properties of the medium change quite smoothly, then the isotropic Riemannian metric is usually used,

$$ds^2 = n^2(x,y)\big(dx^2 + dy^2\big) = \mathrm{e}^{2\mu(x,y)}\big(dx^2 + dy^2\big), \tag{1}$$

the simplest of which arises under the assumption that the speed of propagation of elastic waves increases linearly with depth (a two-dimensional case is considered). Geodesics of such a metric are arcs of circles with centers at points of the line $y = -b/a$. The metric (1), with $n(x,y) = (ay+b)^{-1}$, is sometimes used in the numerical implementation of the procedure for linearization of the inverse kinematic problem of seismics (IKPS).

In mathematical models of 2D tomography, the most common canonical domain is the unit disk. In the disk B we are interested in the metrics of negative and positive curvature. In the upper half-plane we are interested in the metrics of negative curvature.

Currently, numerical modeling and computational simulation within the framework of problems posed in areas containing inhomogeneous complex media are experiencing some difficulties due to the shortage of specific Riemannian metrics with known characteristics that are suitable for use in numerical experiments for research purposes.

A family of metrics are proposed that significantly expand the possibilities of numerical modeling of direct and inverse problems posed in inhomogeneous media. At the first stage, a generalization is made of the metrics of constant curvature in a disk and half-plane. The second stage of the generalization process made it possible to expand the list of suitable Riemannian metrics. Thus, we select and use conformal mapping of the unit disk onto the half-plane and its inverse, which allow metrics constructed in a half-plane to be converted into metrics in a disk, and vice versa.

The geometric characteristics of the proposed families of three-parameter Riemannian metrics, defined in the half-plane and disk, are established; both the original ones and their images under conformal mapping of the areas. These are the components of the metric tensors, the Christoffel symbols, the curvature tensors and the scalar curvature of metrics of variable negative or positive curvatures. The usage of the conformal mappings for transforming the systems of geodesics into each other, led to non-standard formulations of IKPS and inverse problems for inhomogeneous media and, in particular, problems of refraction tomography.

An original interpretation has been proposed for one of the mathematical models of the IKPS, posed in a half-plane, as an external problem of refractive tomography with incomplete data, posed in a disk [2]. The problems of refraction tomography, consisting in the restoration of functions, vector or tensor fields from their ray transforms along geodesics, can be formulated as generalized IKPS in the half-plane for the restoration of scalar, vector or tensor fields. The identified connections between the formulations of IKPS and the problems of refraction tomography can mutually enrich the methods of their research, and obtain solutions to these problems in new original ways.

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STABILITY CONSTANTS FOR DIFFERENTIAL AND DIFFERENCE EQUATIONS WITH DELAY

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Consider a differential equation with concentrated delay

$$\dot{x}(t) = -\sum_{k=1}^{K} a_k(t)x(t - h_k(t)), \quad t \geq 0, \tag{1}$$

and a difference equation

$$x(n+1) - x(n) = -\sum_{k=1}^{K} a_k(n)x(n - h_k(n)), \quad n \in \mathbb{N}_0, \tag{2}$$

which can be considered as a discrete analog to equation (1).

The following two stability conditions for equations (1) and (2) can be considered as generalizations of the "3/2-theorem" by A. D. Myshkis.

Denote $a(t) = \sum\limits_{k=1}^{K} a_k(t)$, $h(t) = \max\limits_{1\leq k\leq K} h_k(t)$.

Theorem 1 [1]. *Suppose $a_k(t) \geq 0$, $h_k(t) \geq 0$ for all $k = \overline{1, K}$. If*

$$\varlimsup_{t\to\infty} \int_{t-h(t)}^{t} a(s)\, ds < \frac{3}{2}, \tag{3}$$

then for the Cauchy function of equation (1) *for some $N, \alpha > 0$ the following estimate is valid:*

$$|C(t,s)| \leq N \exp\left\{-\alpha \int_{s}^{t} a(\tau)\, d\tau\right\}, \quad t \geq s \geq 0. \tag{4}$$

Theorem 2 [2]. *Suppose $a_k(n) \geq 0$, $h_k(n) \geq 0$ for all $k = \overline{1, K}$. If*

$$\varlimsup_{n\to\infty} \sum_{i=n-h(n)}^{n} a(i) < \frac{3}{2}, \tag{5}$$

then for the Cauchy function of equation (2) *for some* $N, \alpha > 0$ *the following estimate is valid:*

$$|C(n,m)| \le N \exp\left\{-\alpha \sum_{i=m}^{n} a(i)\right\}, \quad n \ge m \ge 0. \tag{6}$$

The constant 3/2 is sharp in inequalities (3) and (5): it cannot be reduced without breaking estimates (4) and (6), which is proven by constructing corresponding examples. However, in these examples, there is a significant difference. For differential equations (1), the sharpness of the constant 3/2 is preserved regardless of whether the value $h(t)$ is bounded or not, and for difference equations (2), the sharpness of the constant 3/2 can be proven only if $h(n)$ can take arbitrarily large values. If $h(n)$ is subjected to the boundedness condition, then estimate (5) can be strengthened [3, 4]. A similar effect is observed for semi-autonomous equations: if the coefficients a_k in equations (1) and (2) are constant, then for equation (1) the constant 3/2 remains sharp, while for equation (2) it can be increased [4].

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SOLVABILITY OF NON-STEADY THREE-DIMENSIONAL EQUATIONS OF VISCOUS COMPRESSIBLE HEAT-CONDUCTIVE MULTIFLUIDS

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The system of equations is considered which describes three-dimensional non-steady movements of heat conductive compressible viscous multifluids [1–3]. This model is a generalization of the well-known Navier–Stokes–Fourier model of heat conductive compressible viscous one-component fluids [4, 5] and it includes the laws of conservation of masses, momenta and energy. It is assumed that at every point of space, all components of the multifluid are present, and each has its local velocity, and interaction between them can occur through the exchange of momentum, viscous friction and through the heat exchange.

The characteristic feature of the equations under consideration, in addition to their nonlinearity, is the presence of higher order derivatives of the velocities of all components in the conservation laws for momenta and energy, due to the composite structure of viscous stress tensors, which makes it impossible to automatically expand the theory of heat conductive compressible viscous one-component fluids to the case of multifluids.

This specificity of multicomponent movements can be described using the concept of viscosity matrices. Unlike the one-component case in which the viscosities are scalars, in the multicomponent case they form matrices whose entries are responsible for viscous friction. Diagonal entries stand for viscous friction inside each component, and friction between the components is described by off-diagonal entries. In the case of diagonal viscosity matrices, the equations might interact through lower order terms only. The more complex case of non-diagonal viscosity matrices is looked at in the work. In the general three-dimensional case the theorem is proved for the existence of a weak (dissipative) solution to the initial-boundary value problem describing flows in a bounded domain [6–8].

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ESTIMATES OF SOLUTIONS TO CLASSES OF NONAUTONOMOUS TIME-DELAY SYSTEMS

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Some classes of nonautonomous time-delay systems are considered. We study asymptotic properties of solutions to these systems and obtain estimates characterizing decay rates of the solutions at infinity. Estimates for attraction sets for nonlinear time-delay systems are established. The present work continues our investigations of properties of solutions to nonautonomous time-delay equations (see, for example, [1–3]).

The study was carried out within the framework of the state contract of the Sobolev Institute of Mathematics (project no. FWNF-2022-0008).

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PERIODIC TRAJECTORIES OF A THREE-DIMENSIONAL NONLINEAR CIRCULAR GENE NETWORK MODEL

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We study a three-dimensional dynamic system with nonlinear smooth functions, which simulates a circular gene network with negative feedback:

$$\begin{aligned} \frac{dx_1}{dt} &= L_1(x_3) - \Gamma_1(x_1); \\ \frac{dx_2}{dt} &= L_2(x_1) - \Gamma_2(x_2); \\ \frac{dx_3}{dt} &= L_3(x_2) - \Gamma_3(x_3). \end{aligned} \tag{1}$$

Here and further assume $j = 1, 2, 3$. The smooth monotonically decreasing functions L_j correspond to negative feedback. Degradation of components x_j is described by nonlinear smooth functions Γ_j. As an example, a similar model with linear degradation was given in [1].

We consider the case when for all $j = 1, 2, 3$ there exists an x_j such that

$$\Gamma_j(x_j) = \max L_j(x_{j-1}) = L_j(0).$$

As in [2], the following two lemmas are established.

Lemma 1. *The parallelepiped*

$$\mathcal{Q} = [0, \Gamma_1^{-1}(L_1(0))] \times [0, \Gamma_2^{-1}(L_2(0))] \times [0, \Gamma_3^{-1}(L_3(0))]$$

is invariant for trajectories of the system (1).

Lemma 2. *The system* (1) *has a unique equilibrium point* S_0 *in the interior of* $\mathcal{Q}$.

The construction of an invariant paralelepiped is also described in [3].

Let positive parameters $p_j = \Gamma'_j$, $-q_j = L'_j$ be the derivatives of functions L_j, Γ_j at the point S_0. According to Vyshnegradsky criterion, see [4], if the inequality

$$(p_1 + p_2 + p_3)(p_1p_2 + p_1p_3 + p_2p_3) < p_1p_2p_3 + q_1q_2q_3 \tag{2}$$

holds, then the point S_0 is hyperbolic.

Theorem. *If the condition* (2) *is satisfied then the system* (1) *has a cycle* $\mathcal{C}$ *in the invariand domain* $\mathcal{Q}$*.*

The trajectories of the system (1) lie on a two-dimensional invariant surface.

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SYMPLECTIC AND CONTACT LINEARIZATION OF NONLINEAR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

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The article is considered two differential geometric approaches to solving nonlinear hyperbolic systems in partial differential equations. Such equations are particular case of general Jacobi system [1]. Such systems define a pair of differential 2-forms on the 4-dimensional space $\mathbb{R}^4$. Considered system can be transform to a symplectic Monge–Ampere equation of hyperbolic type. After that we can define conditions under which the Monge–Ampere equation can be transformed to the linear wave equation $u_{tx} = 0$ by a symplectic transformation [1].

Another way is as follows. We consider 5-dimensional 1-jet space $J^1\mathbb{R}^2$ of functions with two independent variables [2]. 1-jet space is provided with a contact structure defined by the Cartan distribution. If Laplace forms are equals zero, the corresponding Monge–Ampere equation can be reduced to the linear wave equation by a contact change of variables [3]. The solution of the wave equation is well known. Applying inverse symplectic or contact transformation to general solution of wave equation, we get the solution of considered nonlinear equation.

Both of these methods will be illustrated using examples of equations arising in the theory of filtration in porous media. Namely, we will consider the problem of deep filtration of a suspension in a porous media [4] and the problem of frontal displacement of oil by a solution of active reagents [5]. For them, conditions for symplectic and contact linearization are found and exact solutions are constructed.

Example. Model of frontal displacement of oil by a solution of carbonized water in large-scale approximation has following form [5]

$$\begin{cases} s_t + H_x = 0, \\ (cs + \phi(1-s) + a)_t + (cH + (1-H)\phi)_x = 0, \end{cases}$$

where $s(t,x)$ is a water saturation, $c(t,x)$ is a concentration of carbon dioxide in water, $H = H(s,c)$ is the Buckley–Leverett function, $a = a(c)$ is a sediment concentration at the pores, $\phi = \phi(c)$ is a carbon dioxide concentration in oil (diffusion), t is time. The x axis coincides with the direction of fluid motion. Here $\phi(c) = c + \delta_1, a(c) = 0$. This system can be reduced to the wave equation by a *symplectic* transformation if and only if

$$H(s,c) = \alpha s + h(c),$$

where α and h are an arbitrary constant and an arbitrary function respectively.

The conditions for the contact equivalence of the equation to the linear wave equation have the form

$$H(s,c) = (\beta s + \gamma)h(c),$$

where β, γ are arbitrary constants and h is an arbitrary function.

Thus, a class of exact solutions to the problem of frontal displacement of oil by carbonated water through symplectic and contact transformation has been obtained.

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ASYMPTOTIC STABILITY OF THE HYBRID SYSTEMS

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A continuous-discrete system of functional-differential equations is a system such that its state is described by two groups of interrelated variables: some variables that are being functions of continuous time satisfy differential equations; others are being functions of discrete time satisfy difference equations. This systems are also called *hybrid*.

Hybrid systems are applied in studying technical objects with impulse and digital control, as well as in economic dynamics modeling [1].

It is natural to construct the solution of hybrid systems step by step, integrating the system on each interval, but one can't study asymptotic properties of every hybrid system solution using this method, so the stability problem for such systems is actual.

Various methods are used in hybrid systems stability studying. The approaches based on the Lyapunov method are applied in papers [1–2], the fixed point principle is applied in paper [3], the Azbelev's W-method is applied in paper [4].

Exact effective coefficient criteria for asymptotic stability can be obtained for hybrid systems such that a continuous subsystem with continuous time is a system of ordinary differential equations [5, 6].

As far as the author of the current paper knows, there are no exact effective coefficient stability criteria for hybrid systems such that the subsystem with continuous time is a system of delay differential equations. Consider the Cauchy problem for an example of the hybrid system of such class

$$\begin{cases} \dot{x}(t) + ax(t-1) = y(n), \quad t \in [n, n+1), \\ x(t) = \psi(t), \quad t \in [-1, 0), \\ y(n) = -bx(n), \\ x(0) = x_0, \end{cases} \qquad n \in \mathbb{N}_0, \tag{1}$$

where $a, b, x_0 \in \mathbb{R}$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, the initial function ψ is assumed to be summable.

System (1) is called asymptotically stable if $\lim_{t\to\infty} x(t) = 0$ for any ψ and x_0.

Let's introduce the operator S that acts in space $C[0,1]$:

$$(Sx)(\tau) = x(1)(1 - b\tau) - a\int_0^\tau x(s)\,ds.$$

Consider the equation

$$\mu - b + (a+b)e^{-\mu} = 0 \tag{2}$$

for the complex variable μ.

Theorem. *Suppose $a \neq 0$. Then the following statements are equivalent:*

- *system (1) is asymptotically stable,*
- *all eigenvalues of the operator S lie inside the unit circle,*
- *the inequality $|\mu| > |a|$ holds for any root of equation (2).*

Corollary. *Suppose $(e^a - 1)/a \leq 2$. The system (1) is asymptotically stable iff $-a < b < a\coth(a/2)$.*

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REPRESENTATIONS OF ALGEBRA $sl_2(\mathbb{R})$ AND ORDINARY DIFFERENTIAL EQUATIONS

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We describe all nonequivalent representations of the algebra $sl_2(\mathbb{R})$ in the space of vector fields $\mathrm{Vect}\,\mathbb{R}^2$ (see [1]). For each of these representations it was found all ordinary differential equations admitting representation data, in terms of a basis differential invariants and operators of the invariant differentiation [2–4]. We also found the Casimir operators of the corresponding universal enveloping algebra, the equations generated by the Casimir operator are integrated and the algebraic independence of the operators of invariant differentiation and Casimir operator are proved.

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ON COMPLEX MONGE–AMPÈRE EQUATION ON POSITIVE CURRENTS OF HIGHER BIDEGREE

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Let M be a complex manifold and T a positive current in M. If u and f are smooth differential forms on M we say that $(\overline{\partial}\partial u)^k = f$ on T if $(\overline{\partial}\partial u)^k \wedge T = f \wedge T$. The question we study in this paper is whether Monge–Ampère equation can be solved on T, and, if so, what kinds of estimates one can find for the solution. Solvability of Monge–Ampère equations in the case $k = 1$ is classical (see [1–3]). The next proposition is important in the proof of the priori inequality for $(\overline{\partial}\partial u)^k$-operator. Let $e_1, \ldots, e_{n+l}$ be a basis for the space of $(1,0)$-forms in $\mathbb{C}^{n+l}$. Write $\gamma = \sum \gamma_{JK} e_J \wedge \overline{e}_K$ and partition γ into a sum $\tau + \sigma$ depending on whether J belongs to K (the τ-part) or not $\gamma = \tau + (\sum_{r=1}^{p-1} \sigma_r + \sigma_0) = \tau + \sigma$.

Proposition. *The quadratic form is defined by $[\gamma, \gamma]\sigma_T = c_{q+p}\gamma \wedge \overline{\gamma} \wedge \omega_{n-q-p} \wedge T$, decompose on positive definite $[\sigma_r, \sigma_r]\sigma_T$ if $(-1)^{p+r} = -1$ and negative definite $[\tau, \tau]\sigma_T$, $[\sigma_r, \sigma_r]\sigma_T$ if $(-1)^{p+r} = 1$, $1 \le r \le p-1$ on the space of primitive forms in $\Lambda_T^{p,q}$. (In the case $p = 0$ form $[\tau, \tau]\sigma_T$ is positive definite, under $p = 2k+1$ form $[\sigma_0, \sigma_0]\sigma_T$ is negative and under $p = 2k$ is positive definite.)*

Proof. Choose a basis $e_1, \ldots, e_{n+l}$ for the space $(1,0)$-forms in $\mathbb{C}^{n+l}$ that diagonalizes both ω and T. Let $dV_j = ie_j \wedge \overline{e}_j$ and $dV_J = \bigwedge_J dV_j$. Then $\omega = \sum dV_j$, $T = \sum \lambda_J dV_J$ and $T \wedge \omega_{n-q-p+1} = \sum_{|K|=n+l-q-p+1}$ $\sum_{1\le i_1<i_2<\ldots<i_{n-q-p+1}\le n+l-q-p+1} \lambda_{K[k_{i_1}k_{i_2}\ldots k_{i_{n-q-p+1}}]} dV_K$ if we let $\lambda_{J[j_{i_1}j_{i_2}\ldots j_{i_{n-q-p+1}}]} = \lambda_{j_1\ldots j_{i_1-1}j_{i_1+1}\ldots j_{i_{n-q-p+1}-1}j_{i_{n-q-p+1}+1}\ldots j_{n+l-q-p+1}}$.

Lemma. *Let $\lambda_I \ge 0$ and presuppose $\sum \lambda_I = 1$. Then* $\sum_{J\cap L=\emptyset} \tau_J \overline{\tau}_L \sum_{1\le i_1<i_2<\ldots<i_{N-l-2p}\le N} \lambda_{(J\cup L)^c[l_{i_1}l_{i_2}\ldots l_{i_{N-l-2p}}]}$ $\le \sum \tau_J \sum_{1\le i_1<i_2<\ldots<i_{N-l-p+1}\le N} \lambda_{J^c[l_{i_1}l_{i_2}\ldots l_{i_{N-l-p+1}}]}|^2$.

Proof. It is sufficient to check the inequality for τ real. We suppose all τ_J pairwise different. The general case follows by continuity. For τ fixed we assume

$$F(\lambda) = F_1(\lambda) +$$

$$+\sum \tau_J \sum_{1\le s_1<\ldots<s_{N-l-p+1}\le N} \sum_{\substack{1\le i_1<\ldots<i_{N-p+1}\le N\\ j_{i_1},\ldots,j_{i_{N-p+1}}[j_{s_1},\ldots,j_{s_{N-l-p+1}}]\cap J\neq\emptyset}}$$

$$\lambda_{j_{i_1}\ldots j_{i_{N-p+1}}[j_{s_1}\ldots j_{s_{N-l-p+1}}]}\times$$

$$\times \sum_{\substack{J\cap L\neq\emptyset\\ L\neq J}} \tau_L \sum_{1\le s_1<\ldots<s_{N-l-p+1}\le N} \sum_{\substack{1\le i_1<\ldots<i_{N-p+1}\le N\\ l_{i_1},\ldots,l_{i_{N-p+1}}[l_{s_1},\ldots,l_{s_{N-l-p+1}}]\cap L\neq\emptyset}}$$

$$\lambda_{l_{i_1}\ldots l_{i_{N-p+1}}[l_{s_1}\ldots l_{s_{N-l-p+1}}]}+$$

$$+\sum_{\substack{J\cap L\neq\emptyset\\ J\neq L}} \tau_J\tau_L(1-\sum_{1\le s_1<s_2<\ldots<s_{N-l-p+1}\le N} \sum_{\substack{1\le i_1<i_2<\ldots<i_{N-p+1}\le N\\ j_{i_1}j_{i_2}\ldots j_{i_{N-p+1}}[j_{s_1}j_{s_2}\ldots j_{s_{N-l-p+1}}]\cap J\neq\emptyset}}$$

$$\lambda_{j_{i_1}j_{i_2}\ldots j_{i_{N-p+1}}[j_{s_1}j_{s_2}\ldots j_{s_{N-l-p+1}}]}-$$

$$-\sum_{1\le s_1<s_2<\ldots<s_{N-l-p+1}\le N} \sum_{\substack{1\le i_1<i_2<\ldots<i_{N-p+1}\le N\\ l_{i_1}l_{i_2}\ldots l_{i_{N-p+1}}[l_{s_1}l_{s_2}\ldots l_{s_{N-l-p+1}}]\cap L\neq\emptyset}}$$

$$\lambda_{l_{i_1}l_{i_2}\ldots l_{i_{N-p+1}}[l_{s_1}l_{s_2}\ldots l_{s_{N-l-p+1}}]})$$

for λ in the simplex $\lambda_I \ge 0$, $\sum \lambda_I = 1$. The proof of the inequality $F_1(\lambda) \ge 0$ comes down to the proof of the inequality

$$\Delta_n^n = \begin{vmatrix} (1-\lambda_1)^2 & \lambda_1\lambda_2 & \ldots & \lambda_1\lambda_{n-1} & \lambda_1\lambda_n \\ \lambda_2\lambda_1 & (1-\lambda_2)^2 & \ldots & \lambda_2\lambda_{n-1} & \lambda_2\lambda_n \\ \ldots & \ldots & \ddots & \ldots & \ldots \\ \lambda_{n-1}\lambda_1 & \lambda_{n-1}\lambda_2 & \ldots & (1-\lambda_{n-1})^2 & \lambda_{n-1}\lambda_n \\ \lambda_n\lambda_1 & \lambda_n\lambda_2 & \ldots & \lambda_n\lambda_{n-1} & (1-\lambda_n)^2 \end{vmatrix} \ge 0,$$

here $\lambda_i \ge 0$, $\sum_{i=1}^n \lambda_i = 1$.

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CRITERION FOR THE SOBOLEV WELL-POSEDNESS OF THE DIRICHLET PROBLEM IN LIPSCHITZ DOMAINS

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Let $G(x,y)$ be the Green function of a bounded Lipschitz domain Ω in $\mathbb{R}^n$, $n \geqslant 2$. The Bogdan formula says that (up to constants)

$$G(x,y) \approx \frac{U(x)U(y)}{\sup_{z\in\Omega:\ |z-(x+y)/2|\leqslant|x-y|} U^2(z)} H(x,y),$$

where $U(x) = \min\{G(x,y_0),1\}$ for some $y_0 \in \Omega$, $\varrho(x) = \operatorname{dist}(x,\partial\Omega)$ and

$$H(x,y) = \begin{cases} |x-y|^{2-n} & \text{for } n \geqslant 3 \quad \text{(see [3])}, \\ \log \frac{\varrho(x)+\varrho(y)+|x-y|e}{|x-y|} & \text{for } n = 2 \quad \text{(see [6])}. \end{cases}$$

This formula is employed to prove the following criterion [6].

Theorem 1. *Put* $W_p^{-1}(\Omega) = (\mathring{W}_q^1(\Omega))'$ *for* $2 \leqslant p = q/(q-1) < \infty$, *with* $\mathring{W}_q^1(\Omega)$ *the closure of* $C_0^\infty(\Omega)$ *in the Sobolev space* $W_q^1(\Omega)$. *The claim*

$$(\forall f \in W_p^{-1}(\Omega))\ (\exists! u \in \mathring{W}_p^1(\Omega)) \quad \Delta u = f \tag{1}$$

is equivalent to the Nyström condition

$$\begin{gathered}(\exists \alpha > 0)\ (\forall a \in \partial\Omega)\ (\forall \beta \in (0, \operatorname{diam}\Omega)) \\ \int_{B(a,\beta)\cap\Omega} (U/\varrho)^p\, dx \leqslant \alpha\beta^{n-p} \sup_{B(a,\beta)\cap\Omega} U^p.\end{gathered} \tag{2}$$

Remark. This criterion is simpler than the similar criterion from [8].

Remark. Condition (2) has emerged in [5] while studying (1) with $W_p^{-1}(\Omega)$ replaced by the Lebesgue space.

Remark. Alkhutov's criterion [1] and many other known facts about (1) may be deduced from Theorem 1 combined with the Carleman–Huber theorem (on the boundary behavior of U) and the property

$$(\forall \tau > -3) \quad \int_\Omega \varrho^\tau U^2\, dx < \infty.$$

Now consider $1 < p < \infty$ and an integer $m \geqslant 2$. In [7] the author has shown that the well-posedness

$$(\forall f \in W_p^{m-2}(\Omega))\ (\exists! u \in W_p^m(\Omega) \cap \mathring{W}_p^1(\Omega)) \quad \Delta u = f \tag{3}$$

implies the condition $\Omega \in \mathcal{W}^{m,p}$ which contains the function U and which is too cumbersome to be exposed here.

Remark. The idea is to combine the Bogdan formula with the straightenability theory of Lipschitz domains due to V. G. Maz’ya and T. O. Shaposhnikova [4] and the author (consult the references in [7]).

Remark. Probably, the converse implication $\Omega \in \mathcal{W}^{m,p} \Rightarrow (3)$ holds if either $m = 2$ or $U(x)$ decays no faster than $\varrho^2(x)$ as $x \to \partial\Omega$.

Remark. Both the straightenability theory and condition $\Omega \in \mathcal{W}^{m,p}$ make use of the Hardy inequality over the family of all dyadic cubes. This is a particular case of the Hardy inequality on trees. The most important criteria for this inequality on trees are given in [2, Theorems 3 and 4]. The author’s paper dedicated to these matters is accepted in Ufa Mathematical Journal.

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DELAY DIFFERENTIAL EQUATIONS AND MATHEMATICAL MODELS IN IMMUNOLOGY

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Mathematical models in immunology are represented by a wide set of differential equations, including delay differential equations. Currently, a significant number of various mathematical models in immunology have been created, in particular: G. Bell, 1970–1978, G. I. Marchuk, 1975–1991, C. Bruni, 1975–1978, A. S. Perelson, 1993–2006, H. T. Banks, 2003–2007, G. A. Bocharov, 2000–2022, et al. Delay differential equations take into account the background of the processes describing the appearance of certain components of the immune response: various cells, antibodies, viral particles, etc. One of the actively developed areas is related to the construction and research of mathematical models of HIV-1 infection, Covid-19 infection, viral hepatitis and other socially significant diseases.

Many mathematical models in immunology can be represented as a system of differential equations of the following form:

$$\frac{dx_i(t)}{dt} = f_i(t, x_t) - (\mu_i + g_i(t, x_t))x_i(t),\ t \geqslant 0, \tag{1}$$

$$x_i(t) = \psi_i(t),\ t \in I_\omega = [-\omega, 0],\ 1 \leqslant i \leqslant m, \tag{2}$$

where $x_t(\theta) = x(t+\theta)$, $\theta \in I_\omega$, $t \geqslant 0$, $f_i(t, x_t)$ is the rate of appearance of i-type elements (cells, viral particles, antibodies, etc.), $\mu_i + g_i(t, x_t)$ is the intensity of the death of i-type elements or their transformation into another element, $\psi_i(t)$ is the number of initial i-type elements, constants $\mu_i > 0$, functions $\psi_i(t)$ are continuous and non-negative, $1 \leqslant i \leqslant m$. The mappings $f_i(t, z)$, $g_i(t, z)$ have the following basic properties: 1) for some constants $a_1 < 0, \ldots, a_m < 0$

$$f_i,\ g_i : R_+ \times C(I_\omega, [a_1, \infty) \times \cdots \times [a_m, \infty)) \to R$$

are continuous, 2) f_i, $g_i : R_+ \times C(I_\omega, R_+^m) \to R_+$, $1 \leqslant i \leqslant m$. For the system (1), (2), the conditions of global solvability and non-negativity of the solution for non-negative initial data are established.

One of the most important problems in studying the solutions of the system (1), (2) is the analysis of the stability of equilibrium positions. In some cases, systems of linear differential equations

$$\frac{dx(t)}{dt} = \sum_{i=0}^{n} C_i x(t-\omega_i) + \int_{-\tau}^{0} C_{n+1}(\theta) x(t+\theta) d\theta, \qquad (3)$$

often arise, where $x(t) = (x_1(t), \ldots, x_m(t))^T \in R^m$, C_i are $m \times m$ matrices, $0 \leqslant i \leqslant n$; $C_{n+1}(\theta)$ is $m \times m$ matrix with Riemann integrable elements, constants $\omega_0 = 0$, $0 < \omega_i < \infty$, $1 \leqslant i \leqslant n$, $0 \leqslant \tau < \infty$.

Some of the systems (3) can be presented in block form

$$\frac{dz(t)}{dt} = Qz(t) + \sum_{i=0}^{n} D_i y(t-\omega_i) + \int_{-\tau}^{0} D_{n+1}(\theta) y(t+\theta) d\theta, \qquad (4)$$

$$\frac{dy(t)}{dt} = \sum_{i=0}^{n} A_i y(t-\omega_i) + \int_{-\tau}^{0} A_{n+1}(\theta) y(t+\theta) d\theta - By(t), \qquad (5)$$

where $x(t) = (z_1(t), \ldots, z_\ell(t), y_1(t), \ldots, y_k(t))^T$, $\ell + k = m$,

$$z(t) = (z_1(t), \ldots, z_\ell(t))^T, \; y(t) = (y_1(t), \ldots, y_k(t))^T,$$

Q is $\ell \times \ell$ stable matrix; $D_0, D_1, \ldots, D_n$ are $\ell \times k$, $A_0, A_1, \ldots, A_n$ are $k \times k$ matrices; $D_{n+1}(\theta)$ is $\ell \times k$, $A_{n+1}(\theta)$ is $k \times k$ matrices, containing Riemann-integrable elements; $B = \operatorname{diag}(b_{11}, \ldots, b_{kk})$, $b_{11} > 0, \ldots, b_{kk} > 0$. The conditions of asymptotic stability or instability of the trivial solution of system (4), (5) written out in terms of non-singular M-matrices, are given.

Two models of the dynamics of HIV-1 infection over a long and relatively short period after human infection are given as examples (high-dimensional non-linear and linear systems of delay differential equations with initial data). The results of analytical research of models solutions are presented. In particular, the asymptotic stability of the equilibrium position interpreted as the absence of HIV-1 infection is investigated. Expressions are given for the indicator R_0 – the basic reproductive number. Analytical studies are supplemented by the results of numerical experiments using an explicit-implicit Euler scheme with a constant integration step.

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BOUNDARY VALUE PROBLEMS FOR FORWARD-BACKWARD PARABOLIC EQUATIONS

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We consider $2n$-parabolic equations with changing time direction. For such problems smoothness of the initial and boundary data does not ensure the smoothness of the solution. An application of the theory of singular equations along with the smoothness of the problem data makes it possible to additionally indicate necessary and sufficient conditions ensuring membership of the solution to the smooth spaces.

In contrast to the classical case, the singular Cauchy operator together with the noncompact integral operators of a special form whose kernels are approximately homogeneous of degree 1 are among these operators. The Fredholm property criterion is established for these operators as well as a formula for the index.

In the article we consider the issues well-posedness of boundary value problems for $2n$-parabolic equations with changing time direction. As it is shown, solutions of boundary value problems depend both on nonintegral Hölder exponent and the coefficients of gluing conditions under necessary and sufficient conditions on input problem data.

SOLUTION OF ONE GENERALIZED LYAPUNOV MATRIX EQUATION

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In the present report, we consider the following matrix equation

$$HA + A^*H - \frac{1}{2p}A^*HA + \frac{1}{4p}(HA^2 + (A^*)^2H) = I, \tag{1}$$

where A is a matrix of size $n \times n$. It is well known that the problem of location of the spectrum of matrix A in the region bounded by a parabola

$$P = \left\{\lambda : (Im\lambda)^2 < 2p\, Re\lambda\right\},\ p > 0,$$

is equivalent to the existence of a solution $H = H^* > 0$ to equation (1) (see, for example, [1, 2]). It follows from Krein's theorem (see [3], Chapter 1) that there is a unique solution to equation (1) and it can be represented as a double contour integral. However, this formula presents difficulties in solving real equations of type (1). In the present report, we give another representation of this solution, which is an analogue of the Lyapunov formula.

Theorem. *Let A be $(n\times n)$-matrix whose eigenvalues are in the domain P, then the solution to matrix equation* (1) *can be represented as:*

$$H = \sum_{k=0}^{\infty}\left(\frac{-1}{4p}\right)^k \left(\int_0^{+\infty}\dots\int_0^{+\infty} e^{-(t_0+\dots+t_k)A^*}\right.$$

$$\left.\circ\left(\sum_{r=0}^{2k} C_{2k}^r(-A^*)^{2k-r}A^r\right)e^{-(t_0+\dots+t_k)A}dt_0\dots dt_k\right).$$

The present report continues the research [4].

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DISTRIBUTED ORDER EVOLUTIONARY EQUATIONS

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Consider the distributed order differentiation operator

$$\mathcal{D}_x^{[\mu]} f(x) = \int D_x^t f(x)\, \mu\,(dt)\,,$$

where D_x^t is a fractional derivative of order t with respect to x, μ is an non-negative Lebesgue–Stieltjes measure, and $\operatorname{supp}\mu \in [0,1)$ and $\sup\operatorname{supp}\mu > 0$.

The report discusses a method for solving initial value problems for a distributed order evolutionary equation of the form

$$\mathcal{D}_x^{[\mu]} u(x) = Lu(x) + f(x), \qquad \lim_{x\to 0} \mathcal{D}_x^{[\mu_1]} u(x) = a. \tag{1}$$

Here L is a linear operator that does not depend on x (it is assumed that $u(x)$, $f(x)$ and a can be elements of some function space, for each fixed x), and μ_1 is the shift of the measure μ by 1. The method makes it possible to construct solutions of the problem (1) in terms of solutions of the problem

$$v'(x) = Lv(x) + g(x), \qquad v(x) = a.$$

The method under discussion is based on an integral transform connecting the operators $\frac{d}{dx}$ and $\mathcal{D}_x^{[\mu]}$. The kernel of the transform is the Wright function with distributed parameters [1]. When μ is concentrated at a point (i.e. $\sup\mu = \{\beta\}$, $\beta \in \mathbb{R}$), the operator $\mathcal{D}_x^{[\mu]}$ coincides (up to a constant factor) with the fractional differentiation operator, and the transform under consideration turns into the Stankovich transform [2, 3].

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POINT SOURCES IDENTIFICATION IN HEAT AND MASS TRANSFER PROBLEMS

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We consider the second order parabolic equation

$$u_t + A(x,D)u = f = \sum_{i=1}^{r} \delta(x-x_i)q_i(t) + f_0,\ (t,x) \in Q = (0,T)\times G, \quad (1)$$

where $A(x,D)u = -\Delta u + \sum_{i=1}^{n} a_i(x)u_{x_i} + a_0(x)u, \quad x \in G \subset \mathbb{R}^n$. The equation (1) is furnished with the initial and boundary conditions

$$u|_{t=0} = u_0,\ \ Bu|_S = g(t,x),\ S = (0,T)\times\Gamma,\ \Gamma = \partial G, \quad (2)$$

where $Bu = u$ or $Bu = \frac{\partial u}{\partial N} + \sigma u = \sum_{i,j=1}^{n} a_{ij}(t,x)u_{x_j}(t,x)\nu_i + \sigma(t,x)u(t,x)$, where $\vec{\nu} = (\nu_1,\dots,\nu_n)$ is the outward unit normal to Γ. The unknowns in (1), (2) are either a solution u and the functions $q_i(t)$ $(i = 1,2,\dots,r)$ occurring into the right-hand side of (1) or a solution u, the functions $q_i(t)$, and the points $\{x_i\}$ $(i = 1,2,\dots,r)$. The overdetermination conditions are as follows:

$$u|_{x=b_i} = \psi_i(t), \quad i = 1,2,\dots,s, \quad (3)$$

where $\{b_i\}$ is a collection of points in G or on Γ.

These problems arise in ecology and in many other fields; in the former case, u is the pollutant concentration, the points $\{x_i\}$ are locations of sources, and $q_i(t)$ $(i = 1,2,\dots,r)$ are their intensities. The main results are connected with numerical methods of solving the problem. Very often the methods rely on reducing the problem to an optimal control problem and minimization of the corresponding objective functional [1]. Some theoretical results devoted to the problem (1)–(3) are available in [2]–[4]. In the article [4] the Dirichlet data on the lateral boundary are complemented with the Neumann data and these data allow to solve the problem on recovering the number of sources, their locations, and intensities. The model problem (1)–(3) ($G = \mathbb{R}^n$) is considered in [5], where the explicit representation of solutions to the direct problem (the Poisson formula) and an auxiliary variational problem are employed to determine numerically the quantities $\sum_i q_i r_{ij}^l$ (here $q_i(t) = const$ for all i and $r_{ij} = |x_i - b_j|$, $l = 1,2,\dots$).

The quantities found allow to determine the points $\{x_i\}$ and intensities q_i. In the one-dimensional case, uniqueness theorem for solutions to the problem (1)–(3) with $r = 1$ is stated in [3]. Similar results are presented also in [6]. Non-uniqueness examples in the problems of recovering of point sources are presented in [7] and some existence theorems in [8]. We employ the Laplace transform and asymptotic representations of the corresponding elliptic problems with a complex parameter to obtain existence and uniqueness theorems.

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DYNAMICS OF SOLUTION DOMAINS OF ORDINARY DIFFERENTIAL EQUATIONS AND STABILITY OVER FINITE TIME INTERVAL

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The report examines the dynamics of the characteristics of domains and boundaries of domains of solutions of ordinary differential equations (ODEs) in the study of stability on a finite time interval. The solution areas arise due to uncertainty in the parameters of differential equations; only the inequalities that are satisfied by the values of these parameters are known. The existence and uniqueness conditions are satisfied for each parameter value. At the same time, the behavior of the areas of exact solutions and the boundaries of the areas, their uniform boundedness with respect to the set of initial values and the set of disturbing influences on a finite time interval are analyzed. Uniform limitation is the basis of practical (technical) stability. The study of this property of mathematical models of technical systems arose from the needs of engineering problems in the field of machine dynamics, design of automatic control systems, radio engineering, rocket science, etc.

In contrast to the classical formulations of stability according to Lyapunov, problems of motion stability in real systems occur over a finite time interval, and at the same time, the initial and permanent disturbances should not exceed a certain value.

To study the dynamics of solution sets, ODE systems are considered

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0, \quad y, f, y_0 \in R^n, \tag{1}$$

which includes many right-hand sides $f(t, y) \in F(t, y)$ and set of initial data $y_0 \in Y_0$. This is due to many reasons: data errors, multiple values, approximation of the right-hand sides f by functions that are more convenient for calculations, etc. The same reasons serve as the basis for the appearance of many values of the initial data Y_0. It is also possible that, in addition to inaccurately given initial data, the right side of the system is affected by perturbing actions $u(t)$, about which it is only known that $u(t) \in U$. For the problem (1), the set of all solutions is described by the formula

$$Y(t, Y_0) = \left\{ y(t, y_0) : \forall y(t_0) \in Y_0, \forall t \geq 0, \frac{dy}{dt} = f(t, y) + u(t) \right\}.$$

For the problem with perturbing action, the set of all possible solutions (trajectories) will be written as follows:

$$Y(t, Y_0) = \left\{ y(t, y_0) : \forall y(t_0) \in Y_0, \forall u(t) \in U, \forall t \geq 0, \frac{dy}{dt} = f(t, y(t), u(t)) \right\}.$$

The report describes new results of using symbolic-numerical methods [1]–[3] for estimating solution sets to study practical stability. For nonlinear ODE systems that have unique solutions in a certain domain of initial data, the boundaries of the domains of initial data transform into the boundaries of the domains of solutions at each specific moment. The class of such nonlinear ODE systems consists of systems that satisfy the constraints of uniform boundedness of solutions (Lagrange stability). Sets of solutions to ODEs, with initial data belonging to the areas of initial data, have complex boundaries (boundary surfaces in dimension space). For boundaries (surfaces) it is impossible to select function formulas with the help of which it was possible to describe the boundaries. As a preliminary, it is useful to construct a regularization of estimates of the boundaries of solution sets, passing to a linear approximation of the original system. Regularization means finding information about a set of exact solutions. The report provides examples of calculations.

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HYPERBOLIC UNIFIED MODEL OF NEWTONIAN CONTINUUM MECHANICS

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Continuum mechanics includes many different branches, the main of which are: solid mechanics (elasticity, plasticity, damage, fracture, etc.), fluid mechanics (dynamics of viscous and inviscid fluid, dynamics of viscoplastic fluid, multiphase flows, etc.), electrodynamics of moving media, (magnetohydrodynamics (ideal and resistive), electrodynamics of moving dielectrics, electromagnetic waves etc.). All of these branches typically use their own specific governing differential equations to describe their respective processes. The question arises: is it possible to use single system of governing equations to describe the processes of all these different disciplines?

The answer seems to be positive and we present a unified model of continuum mechanics [1–3], which makes it possible to simulate processes in elastic and elastoplastic media, as well as the flow of viscous and inviscid fluids. This model can be coupled with the electromagnetic field and extended to continua in the presence of heat transfer and damage of solids. The governing equations of the unified model belong to the class of hyperbolic thermodynamically compatible systems, that is, the system forms a hyperbolic system of first-order partial differential equations and satisfies the laws of thermodynamics (energy conservation and entropy growth). All abovementioned properties of the unified model allow straightforward application of advanced high-order numerical methods and ensure the reliability of numerical solution.

It can be shown by asymptotyc analysis that relaxation limits for hyperbolic model of viscous heat conductive media for small relaxation times give classical Navier–Stokes and Fourier parabolic model for viscous heat conductive fluid flow.

Further generalization of the unified model can be done for multiphase flows of miscible fluids and immiscible fluids with surface tension [4], as well as for deformed porous media saturated with a compressible fluid [5].

A series of numerical test problems is presented, illustrating the applicability of the model for solving problems from various areas of continuum mechanics.

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ON ESTIMATES OF SOLUTIONS TO SYSTEMS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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Consider the linear autonomous system of functional differential equations

$$\dot{x}(t) + \int_0^\tau dQ(s)x(t-s) = f(t), \quad t \in \mathbb{R}_+, \tag{1}$$

where $\tau \in \mathbb{R}_+$, $Q\colon [0,\tau] \to \mathbb{R}^n$ is a matrix-function of bounded variation, $Q(0) = \Theta$, the integrals are understood in the Riemann–Stieltjes sense, the vector-function $f\colon \mathbb{R}_+ \to \mathbb{R}^n$ is locally integrable. Following [1, p. 9–10], without loss of generality, we assume the initial function to be a part of the external perturbation f. We suppose that a solution to system (1) is a locally absolutely continuous vector-function satisfying (1) almost everywhere. System (1) with a given initial condition $x(0) \in \mathbb{R}^n$ is uniquely solvable and its solution is representable in the form [1, p. 84]:

$$x(t) = X(t)x(0) + \int_0^t X(t-s)f(s)\,ds. \tag{2}$$

The matrix-function $X\colon \mathbb{R} \to \mathbb{R}^{n\times n}$ is called the *fundamental matrix* of system (1). It is uniquely defined as the solution to the matrix equation $\dot{X}(t) + \int_0^\tau dQ(s)X(t-s) = \Theta$, $t \in \mathbb{R}_+$, supplemented by the initial data $X(0) = I$, $X(\xi) = \Theta$ for $\xi < 0$. It follows from (2) that the behavior of any solution to (1) is completely determined by properties of X.

In the matrix Q, separate a part of the elements of the main diagonal and rewrite system (1) in the form

$$\dot{x}(t) + Ax(t) + \int_0^\omega dR(s)x(t-s) = \int_0^\sigma dP(s)x(t-s) + f(t), \quad t \in \mathbb{R}_+. \tag{3}$$

Here $A = \operatorname{diag}\{a_1, \ldots, a_n\}$, $a_k \in \mathbb{R}$; $R(s) = \operatorname{diag}\{r_1(s), \ldots, r_n(s)\}$, $r_k(s)$ are monotone functions; all entries of the matrix-function P are nondecreasing; $\omega, \sigma \in \mathbb{R}_+$.

The system defined by the left-hand side of (3), we call the *comparison system* for system (3). Since the matrices A and $R(s)$ are diagonal, the comparison system can be regarded as the family of independent scalar equations

$$\dot{x}(t) + a_k x(t) + \int_0^{\omega} x_k(t-s)\, dr_k(s) = 0, \quad t \in \mathbb{R}_+, \quad k = \overline{1,n}. \tag{4}$$

The fundamental matrix of the comparison system for system (3) is $X_0(t) = \operatorname{diag}\{x_{01}(t), \ldots, x_{0n}(t)\}$, where x_{0k} are fundamental solutions of (4), and the characteristic function of the comparison system for (3) is $G_0(\gamma) = \operatorname{diag}\{g_1(\gamma), \ldots, g_n(\gamma)\}$, where g_k are those of (4).

Sharp two-sided exponential estimates for the fundamental solutions to scalar equations of the form (4) are obtained in [2, 3]. Suppose that the functions g_k have real roots, for $k = \overline{1,m}$ the functions r_k are nondecreasing and for $k = \overline{m+1,n}$ the functions r_k are nonincreasing. Denote by ζ_k the least positive roots of the functions g_k. Using [2, 3], we get

$$\Theta \le X_0(t) \le N e^{-\zeta_0 t}, \tag{5}$$

where $\zeta_0 = \min\{\zeta_1, \zeta_2, \ldots, \zeta_n\}$, $N = \operatorname{diag}\{-\frac{1}{g_1'(\zeta_1)}, \ldots, -\frac{1}{g_m'(\zeta_m)}, 1, \ldots, 1\}$.

Let $H(\gamma) = I - G_0^{-1}(\gamma)\left(G_0(\gamma) - \int_0^{\sigma} e^{\gamma s} dP(s)\right)$.

Theorem. *Suppose that the fundamental matrix of the comparison system satisfies estimates* (5), *the matrix* $H(0)$ *is positively invertible, and* γ_0 *is the first positive root of the equation* $\det H(\gamma) = 0$. *Then for all* $\gamma \in [0, \gamma_0)$, *the fundamental matrix of system* (1) *admits the two-sided estimate* $\Theta \le X(t) \le H^{-1}(\gamma) N e^{-\gamma t}$, $t \in \mathbb{R}_+$.

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DEFINITION AND SOME PROPERTIES OF MEASURES OF STABILITY AND INSTABILITY OF A DIFFERENTIAL SYSTEM

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For a given $n \in \mathbb{N}$ and zero neighborhood $G \subset \mathbb{R}^n$, we consider the differential system

$$\dot{x} = f(t, x), \quad f(t, 0) \equiv 0, \quad t \in \mathbb{R}_+ \equiv [0, +\infty), \quad x \in G, \tag{1}$$

where $f, f'_x \in C(\mathbb{R}_+, G)$. Let's put

$$B_\delta \equiv \{x_0 \in \mathbb{R}^n \mid 0 < |x_0| < \delta\}, \quad \Delta \equiv \sup\{\delta \mid B_\delta \subset G\},$$

and denote by $x(\cdot, x_0)$ a non-extendable solution of system (1) with the initial value $x(0, x_0) = x_0$.

The differential system (1) is completely deterministic, however, it is possible to give a natural stochastic meaning to its measures of stability $\mu_\varkappa(f)$ or instability $\nu_\varkappa(f)$ [1, 2]. They allow us to estimate from below the possibility or impossibility of randomly selecting the initial value x_0 of perturbed solution $x(\cdot, x_0)$, arbitrarily close to zero, so that its graph falls into a given tube of the zero solution in any of the following senses [3, 4]:

a) immediately on the entire time semi-axis (the Lyapunov stability for $\varkappa = \lambda$);

b) at least episodically, but at arbitrarily late points in time (the Perron stability for $\varkappa = \pi$);

c) at least from some moment, but then forever (the upper-limit stability for $\varkappa = \sigma$).

The forerunners of the described measures were the recent concepts of almost stability and almost complete instability [5], which provide the corresponding properties of solutions with a full measure.

Definition 1. We will say that system (1) has the following property of the *Lyapunov, Perron* or *upper-limit* type:

1) *stability* (*almost stability*) if for any $\varepsilon > 0$ there exists $\delta \in (0, \Delta)$, such that any (respectively, almost any in the sense of the Lebesgue measure) initial value $x_0 \in B_\delta$ satisfies the corresponding requirement

$$\sup_{t\in\mathbb{R}_+} |x(t,x_0)| < \varepsilon, \qquad \varliminf_{t\to+\infty} |x(t,x_0)| < \varepsilon, \qquad \varlimsup_{t\to+\infty} |x(t,x_0)| < \varepsilon; \tag{2}$$

2) *complete instability* (*almost complete instability*) if there exist $\varepsilon > 0$ and $\delta \in (0, \Delta)$, such that any (respectively, almost any) initial value $x_0 \in B_\delta$ does not satisfy the corresponding requirement (2) (which is considered to be unfulfilled by definition, in particular, when the solution $x(\cdot, x_0)$ is not defined on the entire ray $\mathbb{R}_+$).

Definition 2. For system (1), the number

$$\mu_\varkappa(f) \in [0,1], \quad \varkappa = \lambda, \pi, \sigma,$$

is called, respectively, *the Lyapunov, Perron* and *upper-limit measure of stability*, if system (1):

a) for each $\mu < \mu_\varkappa(f)$ is *μ-stable*, i.e. for any $\varepsilon > 0$ there exists $\delta_\varepsilon \in (0, \Delta)$, such that for every $\delta \in (0, \delta_\varepsilon)$ all values $x_0 \in B_\delta$, satisfying the corresponding requirement (2), form a subset, which relative measure (in the Lebesgue sense) in B_δ is

$$\mathrm{M}_\varkappa(f, \varepsilon, \delta) \geq \mu;$$

b) for each $\mu > \mu_\varkappa(f)$ is not μ-stable.

Definition 3. For system (1), the number

$$\nu_\varkappa(f) \in [0,1], \quad \varkappa = \lambda, \pi, \sigma,$$

is called, respectively, *the Lyapunov, Perron* and *upper-limit measure of instability*, if system (1):

a) for each $\nu < \nu_\varkappa(f)$ is *ν-unstable*, i.e. for any $\varepsilon > 0$ there exists $\delta_\varepsilon \in (0, \Delta)$, such that for every $\delta \in (0, \delta_\varepsilon)$ all values $x_0 \in B_\delta$, unsatisfying the corresponding requirement (2), form a subset, which relative measure (in the Lebesgue sense) in B_δ is

$$\mathrm{N}_\varkappa(f, \varepsilon, \delta) \geq \nu;$$

b) for each $\nu > \nu_\varkappa(f)$ is not ν-unstable.

The correctness of Definitions 2 and 3 is justified by the following theorems.

Theorem 1. *For any system* (1), *any* $\varepsilon > 0$ *and each of the requirements* (2), *the sets of all points* $x_0 \in G$, *both satisfying this requirement and not satisfying it, are measurable.*

Theorem 2. *For any system* (1) *the set of all values* $\mu \in [0, 1]$ *for which it is Lyapunov, Perron or upper-limit* μ*-stable, as well as all values* $\nu \in [0, 1]$, *for which it is* ν*-unstable, obviously contains the point 0 and represents an interval, possibly degenerate to this point.*

The following two theorems offer specific formulas for measures of stability and instability and define a set of basic relations linking various measures.

Theorem 3. *For each system* (1), *the entire six of its Lyapunov, Perron and upper-limit measures of stability or instability are uniquely defined, which are respectively given by the formulas*

$$\mu_{\varkappa}(f) = \lim_{\varepsilon\to+0} \varliminf_{\delta\to+0} \mathrm{M}_{\varkappa}(f, \varepsilon, \delta), \quad \nu_{\varkappa}(f) = \lim_{\varepsilon\to+0} \varliminf_{\delta\to+0} \mathrm{N}_{\varkappa}(f, \varepsilon, \delta), \qquad (3)$$

where the limits at $\varepsilon \to +0$ *can be replaced by the lower or, respectively, upper exact bound on* $\varepsilon > 0$.

Theorem 4. *For any system* (1) *the inequalities are satisfied*

$$0 \leqslant \mu_{\lambda}(f) \leqslant \mu_{\sigma}(f) \leqslant \mu_{\pi}(f) \leqslant 1, \quad 0 \leqslant \nu_{\pi}(f) \leqslant \nu_{\sigma}(f) \leqslant \nu_{\lambda}(f) \leqslant 1, \qquad (4)$$

$$0 \leqslant \mu_{\varkappa}(f) + \nu_{\varkappa}(f) \leqslant 1. \qquad (5)$$

Almost stability and almost complete instability are naturally associated with single values of the corresponding measures, but this logical connection turns out to be only one-way.

Theorem 5. *System* (1) *has almost stability or almost complete instability (of some type) if and only if it is 1-stable or, accordingly, 1-unstable (of that type), and then its measures of stability and instability (of the same type) are equal to 1 and 0 or, respectively, vice versa.*

Theorem 6. *For* $n = 2$, *there are two autonomous systems of the form* (1), *which have neither almost stability nor almost complete instability of any of the three types: one of them has measures of stability and instability of all three types equal to 1 and 0, respectively, and the other is the opposite.*

In the case of a linear system, the Lyapunov and upper-limit measures can only take their extreme values, which are obviously also realized on the Perron measures — this is what the following two theorems establish.

Theorem 7. *For any linear system* (1), *only the following two situations are possible, and in formulas* (3) *for all measures of stability and instability mentioned in them, the lower limits for* $\delta \to +0$ *are exact:*

1) *either the relations are satisfied*

$$\mu_\lambda(f) = \mu_\sigma(f) = \mu_\pi(f) = 1 > 0 = \nu_\pi(f) = \nu_\sigma(f) = \nu_\lambda(f)$$

and system (1) *has stability of all three types;*

2) *either the relations are satisfied*

$$\mu_\lambda(f) = \mu_\sigma(f) = 0 < 1 = \nu_\sigma(f) = \nu_\lambda(f)$$

and system (1) *has the Lyapunov and upper-limit almost complete (possibly even complete) instability.*

In addition, in the linear case, the upper-limit complete instability follows from the Lyapunov one, but the Perron instability does not follow, and not to any extent.

Theorem 8. *For any* $n \in \mathbb{N}$, *each of the situations listed in Theorem 7 is realized on some limited scalar linear system of the form* (1), *and the second situation is realized on at least two systems: one of them is autonomous and has the Perron complete instability, i.e.*

$$\mu_\pi(f) = 0 < 1 = \nu_\pi(f),$$

and the other — the Perron stability, i.e.

$$\mu_\pi(f) = 1 > 0 = \nu_\pi(f).$$

The set of all possible sets of different measures of stability and instability of one-dimensional systems is finite.

Theorem 9. *For* $n = 1$, *the measures of stability and instability of any system* (1) *satisfy the relations*

$$\mu_\lambda(f) = \mu_\sigma(f) \leqslant \mu_\pi(f), \quad \nu_\pi(f) \leqslant \nu_\sigma(f) = \nu_\lambda(f), \tag{6}$$

$$\mu_\varkappa(f), \nu_\varkappa(f) \in \{0,\ 1/2,\ 1\}, \quad \mu_\varkappa(f) + \nu_\varkappa(f) = 1, \quad \varkappa = \lambda, \pi, \sigma. \tag{7}$$

Theorem 10. *For* $n = 1$, *both inequalities in chains* (6) *for some limited linear system* (1) *are strict, and the cases of all equalities in these chains* *for each pair of measures of stability and instability specified by conditions* (7) *are implemented on some autonomous systems* (1).

Theorem 6 simultaneously confirms the realizability of both zero and one values by all measures of stability or instability for two-dimensional autonomous systems. Moreover, for such systems the set of implementable sets of all measures turns out to be quite rich.

Theorem 11. *For $n = 2$, for each individual non-strict inequality in chains* (4) *and* (5) *there are two autonomous systems of the form* (1)*: for one of them it turns into an equality, and for the other into a strict inequality.*

Theorem 12. *For $n = 2$, for any $r > 0$ there exists an autonomous system* (1)*, in which the measures of stability of all three types take the same positive value, as well as all measures of instability, and the ratio of these two values equals r, and the right inequality in chain* (5) *turns into equality.*

The following two theorems implement the most contrasting situations in the autonomous arbitrarily non-one-dimensional case.

Theorem 13. *For every integer $n > 1$, some autonomous system* (1) *satisfies the relations*

$$\mu_\lambda(f) = \mu_\sigma(f) = 0 < 1 = \mu_\pi(f), \quad \nu_\pi(f) = \nu_\sigma(f) = 1 > 0 = \nu_\lambda(f).$$

Theorem 14. *For every integer $n > 1$, some autonomous system* (1) *satisfies the relations*

$$\mu_\lambda(f) = 0 < 1 = \mu_\sigma(f) = \mu_\pi(f), \quad \nu_\pi(f) = 1 > 0 = \nu_\sigma(f) = \nu_\lambda(f).$$

In the one-dimensional autonomous case, two contrasting situations described in Theorems 13 and 14 are impossible.

Theorem 15. *For $n = 1$, for any autonomous system* (1) *the equalities are satisfied*

$$\mu_\lambda(f) = \mu_\sigma(f) = \mu_\pi(f), \quad \nu_\pi(f) = \nu_\sigma(f) = \nu_\lambda(f).$$

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RECONSTRUCTION OF A NONLINEAR COEFFICIENT IN SORPTION MODEL WITH GAS DIFFUSION

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The model of the gas absorption process in the sorption column, taking into account the diffusion in the gas flow, has the form

$$u_t + a_t + \nu u_x = D u_{xx}, \qquad 0 < x < l, \qquad 0 < t \leqslant T, \tag{1}$$

$$a_t = \varphi(u) - a, \qquad 0 < x < l, \qquad 0 < t \leqslant T, \tag{2}$$

$$u(0,t) = \mu(t), \qquad u(l,t) + \lambda\, u_x(l,t) = 0, \qquad 0 \leqslant t \leqslant T, \tag{3}$$

$$a(x,0) = 0, \qquad u(x,0) = 0, \qquad 0 \leqslant x \leqslant l, \tag{4}$$

where D – diffusion coefficient; λ – proportionality coefficient between the intensity of gas flow in the right extreme section of the tube and the difference in gas concentration in the right section of the tube and outside the right end of the tube; the function $u(x,t)$ determines the concentration (density) of gas in the section x of the sorption tube (column) at time t; the function $a(x,t)$ determines the concentration of gas in the sorbent grains located inside the tube in the section x at time t; the function $\mu(t)$ sets the concentration of gas in the flow at the inlet to the tube at $x = 0$; the function $\varphi(s)$ is the sorption isotherm, indicating the ratio between the gas densities in the pores and in the sorbent grains.

In a direct problem (1)–(4), it is required to determine the functions $u(x,t)$, $a(x,t)$ by given positive values l, T, D, λ and given functions $\mu(t)$, $\varphi(s)$.

In the inverse problem, by known values l, T, D, λ, the known function $\mu(t)$ and additional given function $h(t)$, such that

$$h(t) = u_x(0,t) \qquad \forall t \in [0,T], \tag{5}$$

it is required to define sorption isotherm $\varphi(s)$ and functions $u(x,t)$, $a(x,t)$.

The solvability of problem (1)–(4) are proposed by A. M. Denisov, S. R. Tuikina and A. Lorenzi [1, 2] in the form of conditions on given functions:

$$\mu(t) \in C^1[0,T], \quad 0 < \mu_o \leqslant \mu'(t) \leqslant \mu_1 \quad \forall\, t \in [0,T], \quad \mu(0) = \mu'(0) = 0, \tag{6}$$

$$\varphi(s), \varphi'(s) \in C(-\infty,\infty), \quad 0 \leqslant \varphi'(s) \leqslant \varphi_0 \quad \forall\, s \in (-\infty,\infty), \quad \varphi(0) = 0. \qquad (7)$$

When conditions (6), (7) are met, the unique solution of the direct problem (1)–(4) exists as functions $u(x,t)$, $a(x,t) \in C^{2,1}\big(\bar{Q}_{l,T}\big)$; such that $u_t(x,t) > 0$, $a_t(x,t) > 0$, $\forall x \in [0,l]$ $\forall t \in [0,T]$; $0 \leqslant u(x,t) \leqslant \mu(\tau)$, $0 \leqslant a(x,t) \leqslant \varphi\big(\mu(\tau)\big)$, $\forall (x,t) \in \overline{Q}_{l,\tau}$ $\forall \tau \in [0,T]$, where $Q_{l,T} = \{(x,t):\ 0 < x < l,\ 0 < t \leqslant T\}$.

We consider the possibility of obtaining the solution of problem (1)–(4) from the integral equation

$$u(x,t) = \Big(1 - \frac{x}{l+\lambda}\Big)\,\mu(t) + \sum_{n=1}^{+\infty} \frac{2}{l + \big(\cos\big(\sqrt{\omega_n^*}\,l\big)\big)^2\,\widehat{\lambda}} \int_0^t e^{-(D\omega_n^* + \beta)(t-\tau)} \times$$

$$\times \int_0^l e^{\frac{\nu}{2D}\,(x-s)} \Big[\frac{\nu}{l+\lambda}\mu(\tau) - \Big(1 - \frac{s}{l+\lambda}\Big)\mu'(\tau) - \varphi\big(u(s,\tau)\big) +$$

$$+ \int_0^\tau e^{-(\tau-\theta)} \varphi\big(u(s,\theta)\big)\,d\theta\Big] \sin\big(\sqrt{\omega_n^*}\,s\big)\,ds\,d\tau\, \sin\big(\sqrt{\omega_n^*}\,x\big),\ (x,t) \in \overline{Q}_{l,T}, \qquad (8)$$

in which the values of ω_n^* are calculated from the algebraic equation

$$\sqrt{\omega} = -\frac{1}{\widehat{\lambda}}\,\mathrm{tg}\,\big(\sqrt{\omega}\,l\big)\,. \qquad (9)$$

After determining the solution $u(x,t)$ of equation (8), the function $a(x,t)$ is calculated by the formula

$$a(x,t) = \int_0^t e^{-(t-\tau)} \varphi\big(u(x,\tau)\big)\,d\tau, \qquad (x,t) \in \overline{Q}_{l,T}. \qquad (10)$$

The equations (8), (9) and formula (10) can be used for recovering the solution of the inverse problem (1)–(5).

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INVERSE PROBLEM FOR A POPULATION MODEL WITH NONLOCAL COEFFICIENTS

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The mathematical model of population dynamics used here in the form of a direct problem is presented in the books [1] (p. 153–155) and [2] (p. 160) as an initial boundary value problem for a nonlinear PDE of I order:

$$u_x(x,t)+u_t(x,t) = -\,\mu_0(x)u(x,t)-\mu_1(x)\Psi(S(t))u(x,t), \quad (x,t)\in\overline{Q}_T, \quad (1)$$

$$u(0,t) = \Phi(S(t))\int_0^l \beta(\xi)u(\xi,t)\,d\xi, \quad S(t) = \int_0^l \gamma(\xi)u(\xi,t)\,d\xi, \quad t\in[0,T], \quad (2)$$

$$u(x,0) = \varphi(x), \qquad x\in[0,l], \quad (3)$$

where $Q_T = \{(x,t):\ 0<x\leqslant l,\ 0<t\leqslant T\}$, and $\overline{Q}_T$ is closing the domain Q_T. The function $u(x,t)$ determines the number of individuals of age x in the population at time t; the functions $\mu_0(x)$ and $\mu_1(x)$ characterize the intensity of mortality of individuals from natural causes and from overpopulation, respectively; the function $\beta(x)$ is the density of reproduction of individuals with zero age from a parent of age x; $S(t)$ is the total number of individuals at time t, taking with a density of $\gamma(x)$; $\Psi(s)$ is an indicator of the effect of the total number of individuals on additional mortality from overpopulation; $\Phi(s)$ is an indicator of the influence of the total population on fertility; $\varphi(x)$ is initial distribution of individuals.

In a direct problem, it is required to determine the function $u(x,t)$ by given values l, T and functions $\mu_0(x)$, $\mu_1(x)$, $\Psi(s)$, $\Phi(s)$, $\beta(x)$, $\gamma(x)$, $\varphi(x)$.

In the inverse problem (IP), by given $l>0$ and $T\geqslant l$, it is required to define functions $\mu_1(x)$ and $u(x,t)$ by known $a\in(0,l]$ for a known function

$$g(t) = u(a,t), \qquad t\in[0,l],$$

given in addition to functions $\mu_0(x)$, $\Psi(s)$, $\Phi(s)$, $\beta(x)$, $\gamma(x)$, $\varphi(x)$.

The beginning of mathematical studies of biological populations with growth of organisms, can be attributed to the paper of L. Euler [3], the research is continued by A. Lotka in a large cycle of works (see, for example,

[4], [5]), and later by a lot of authors [1], [2]. Identification of parameters in models of age-structured populations dynamics in the form of inverse problems has been developed in XXI century (see, for example [6], [7]).

The solvability of the direct and the uniqueness for the inverse problems are presented under assumptions of the considered coefficients' nonnegativity, which fully corresponds to concepts of the modeled process.

Theorem 1. *If the following conditions:* $l \leqslant T$, $\mu_1(x), \varphi(x) \in C^1[0,l]$; $\mu_0(x), \beta(x), \gamma(x) \in C[0,l]$; $\mu_0(x), \mu_1(x), \beta(x), \gamma(x) \geqslant 0$, $\varphi(x) > 0$ $\forall x \in [0,l]$ $\forall t \in [0,T]$; *are met with conditions*

$$\varphi(0) = \Phi(S(0)) \int_0^l \beta(\xi)\, \varphi(\xi)\, d\xi; \tag{4}$$

$$\Phi(s), \Psi(s) \in C^1(\mathbb{R}), \qquad \exists \Phi_0, \Phi_1, L_\Phi\colon\ 0 \leqslant \Phi(s),\ \Psi(s) \leqslant \Phi_0; \tag{5}$$

$$|\Phi'(s)|,\ |\Psi'(s)| \leqslant \Phi_1; \quad |\Phi'(s) - \Phi'(\xi)|,\ |\Psi'(s) - \Psi'(\xi)| \leqslant L_\Phi |s - \xi|\ \forall s, \xi \in \mathbb{R}, \tag{6}$$

then there is a unique solution $u(x,t) \in C^1(\overline{Q}_T)$ *of the problem* (1)–(3).

Theorem 2. *If* a, l, T, $\mu_0(x)$, $\Psi(s)$, $\Phi(s)$, $\beta(x)$, $\gamma(x)$, $\varphi(x)$, $g(t)$ *satisfy conditions* (4)–(6), $a = l > 0$, $T \geqslant l$, $g(t) \in C^1[0,l]$; $\beta(x), \gamma(x) \in C[0,l]$; $\mu_0(x), \varphi(x) \in C^1[0,l]$; $\beta(x)$, $\gamma(x)$, $\mu_0(x) \geqslant 0$, $\varphi(x)$, $g(t) > 0$ $\forall x, t \in [0,l]$; $g(0) = \varphi(a)$, *then inverse problem* (IP) *can have no more than one solution.*

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AN INITIAL-BOUNDARY VALUE PROBLEM FOR A PSEUDOHYPERBOLIC EQUATION

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We consider the differential equation

$$\big(I-\Delta\big)D_t^2u+\Delta^2u-a^2\Delta u=f(t,x),\quad t>0,\quad x\in\mathbb{R}^n_+, \tag{1}$$

with initial conditions

$$u|_{t=0}=0,\quad D_tu|_{t=0}=0,$$

and boundary conditions

$$\big(b_{14}D^3_{x_n}u+b_{13}D^2_{x_n}u+b_{12}D_{x_n}u+b_{11}u\big)\big|_{x_n=0}=0,$$

$$\big(b_{24}D^3_{x_n}u+b_{23}D^2_{x_n}u+b_{22}D_{x_n}u+b_{21}u\big)\big|_{x_n=0}=0,$$

where I is the identity operator, Δ is the Laplace operator with respect to the spatial variables $x=(x',x_n)$, $a\in\mathbb{R}$ and $b_{ij}\in\mathbb{R}$, $i,j=1,2$.

Equation (1) belongs to the class of pseudohyperbolic equations. This class was introduced in the monograph [1]. Such equations are usually called Sobolev-type equations after Sobolev's pioneering works [2]. Differential equation (1) arises in modeling torsional [3] or longitudinal [4] vibrations of elastic rods.

Let the Lopatinsky condition hold for boundary value problem with certain set of coefficients b_{ij}. Conditions for the unique solvability of this problem in anisotropic weighted Sobolev spaces are established and estimates of solutions are obtained.

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LIMIT CYCLES OF "NORMAL SIZE" OF LIENARD SYSTEMS OF TYPE 3A+2S

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In the qualitative study of autonomous systems in the plane, the most difficult problem is to estimate the number of limit cycles, which is not solved even for the simplest classes of such systems. In this paper we consider the Lienard system:

$$\frac{dx}{dt} = y, \ \ \frac{dy}{dt} = -g(x) - \epsilon f(x)y, \tag{1}$$

where $f(x)$ is a first degree polynomial and $g(x) = x(1-x)(1-Kx)(1-Lx)(1-Mx)$, $-1 < L < 0$, $0 < K < M < 1$, $\epsilon > 0$ is a small parameter.

System (1) can be represented by known transformations in the form of

$$\frac{dx}{dt} = y - \epsilon F(x), \ \ \frac{dy}{dt} = -g(x). \tag{2}$$

At such parameter values, the system under consideration has three antisaddles and two saddles. The results of the studies [2, 3] can be naturally generalized to the case of five special points.

S. Smale in his paper [4] supported the hypothesis that the Lienard system (1) in the case $g(x) = x$, and $F(x)$ is a polynomial of degree $2k+1$ and $F(0) = 0$, can have at most k limit cycles around the antisaddle $O(0,0)$.

Hypothesis. In the parameter space of the system (2) with $g(x) = x$ there exists a region Ω in which the number of limit cycles of the system (1) does not exceed the number m of zeros of the odd part of the function $F(x)$, i.e., the positive zeros of the function

$$\varphi(x) = F(x) - F(-x)$$

and also inside Ω there exists a subarea in which this number is equal to m.

The Lienard system (2) is remarkable in that all its special points belong to the Ox axis.

Definition. Let the Lienard system (1) have an antisaddle $A(x_0, 0)$. Let us denote, by ξ_1 (ξ_2), the abscissa of the special point nearest to the left (right) of A; if there are no special points on the left (right), we consider $\xi_1 = -\infty$ ($\xi_2 = +\infty$). We will call the prediction system around the special point $A(x_0, 0)$ for the Lienard system (2) a system of

$$F(\eta) = F(\mu), \ \ G(\eta) = G(\mu),$$

where $F(\eta) = \int_{x_0}^{\eta} f(x)dx$, $G(\eta) = \int_{x_0}^{\eta} g(x)dx$, $\xi_1 < \eta < x_0$, $x_0 < \mu < \xi_2$.

The same method is used to construct the systems (1) with the distributions (((1,0),1),1), (((0,0),1),1), (((0,0),2),0), (((0,0),0),2) of limit cycles of "normal size".

To accurately estimate the number of limit cycles, we will use the Dulac–Cherkas function [1].

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APPLICATION OF HILBERT RESOLVENT FORMULA TO EVALUATION OF INTEGRALS WITH SPECIAL FUNCTION KERNELS

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Let consider two important well-known facts from quiet different fields of mathematics. The first fact is the Hilbert resolvent identity from functional analysis [1]

$$R_\lambda(A) - R_\mu(A) = (\lambda - \mu) R_\lambda(A) R_\mu(A), \tag{1}$$

here A is a linear bounded operator on Banach space, λ and μ are any complex numbers, $R_\lambda(A)$ and $R_\mu(A)$ are resolvents of A with spectral parameters λ, μ.

The second fact is a core identity for a resolvent of Riemann–Liouville fractional integral

$$R_\lambda f(x) = -\frac{1}{\lambda} f(x) - \frac{1}{\lambda^2} \int\limits_a^x E_{\alpha,\alpha}\left(\frac{(x-y)^\alpha}{\lambda}\right) (x-y)^{\alpha-1} f(y)\, dy, \tag{2}$$

here $E_{\alpha,\beta}$ is the Mittag–Leffler function [2–5]. This formula is due to Hille and Tamarkin.

The main idea of this research is to substitute the resolvent of Riemann–Liouville fractional integral (2) into the abstract Hilbert resolvent identity (1). In this way it is possible to evaluate rather sophisticated integrals from products of Mittag–Leffler functions. This is a new method for evaluating integrals of such type.

As an example we present one of similar results concerning evaluation of integrals with the product of Mittag–Leffler functions.

Theorem. *For resolvents of Riemann–Liouville fractional integrals* (2) *the next formula is valid as derived from the Hilbert resolvent identity* (1)*:*

$$\int\limits_t^x (y-t)^{\alpha-1} (x-y)^{\alpha-1} E_{\alpha,\alpha}\left(\frac{(y-t)^\alpha}{\mu}\right) E_{\alpha,\alpha}\left(\frac{(x-y)^\alpha}{\lambda}\right) dy$$

$$= \frac{\mu\lambda}{\lambda-\mu} (x-t)^{\alpha-1} \left(E_{\alpha,\alpha}\left(\frac{(x-t)^\alpha}{\mu}\right) - E_{\alpha,\alpha}\left(\frac{(x-t)^\alpha}{\lambda}\right)\right).$$

As consequences we list the next integral formulas.

For $\alpha = 1$ an obvious formula is valid

$$\int_t^x e^{\frac{y-t}{\mu}} e^{\frac{x-y}{\lambda}} dy = \frac{\mu\lambda}{\lambda - \mu} \left(e^{\frac{x-t}{\mu}} - e^{\frac{x-t}{\lambda}} \right).$$

For $\alpha = 2$ the next formula is valid

$$\int_t^x \sinh\left(\frac{y-t}{\sqrt{\mu}}\right) \sinh\left(\frac{x-y}{\sqrt{\lambda}}\right) dy$$

$$= \frac{\sqrt{\lambda\mu}}{\lambda - \mu} \left(\sqrt{\mu} \sinh\left(\frac{x-t}{\sqrt{\mu}}\right) - \sqrt{\mu} \sinh\left(\frac{x-t}{\sqrt{\lambda}}\right) \right).$$

For $\alpha = 3$ the next formula is valid

$$\int_t^x (y-t)^2 (x-y)^2 \, {}_0F_2\left(; \frac{4}{3}, \frac{5}{3}; \frac{(y-t)^3}{27\mu} \right) {}_0F_2\left(; \frac{4}{3}, \frac{5}{3}; \frac{(x-y)^3}{27\lambda} \right) dy$$

$$= \frac{2\mu\lambda}{\lambda - \mu} (x-t)^2 \left({}_0F_2\left(; \frac{4}{3}, \frac{5}{3}; \frac{(x-t)^3}{27\mu} \right) - {}_0F_2\left(; \frac{4}{3}, \frac{5}{3}; \frac{(x-t)^3}{27\lambda} \right) \right),$$

where ${}_0F_2$ is the Gauss hypergeometric function.

The method may be also applied for more resolvents and special function kernels. Further examples involve other types of fractional integrals, general integral operators, resolvents from applied problems of physics and mechanics.

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MULTI-DIMENSIONAL INTEGRAL TRANSFORMS WITH H-FUNCTION IN THE KERNEL IN THE WEIGHTED SPACES OF SUMMABLE FUNCTIONS

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Multidimensional integral transform

$$\big(\mathrm{H}f\big)(\mathbf{x}) = \int\limits_{\mathbf{0}}^{\infty} \mathrm{H}^{\mathbf{m},\,\mathbf{n}}_{\mathbf{p},\,\mathbf{q}}\left[\mathbf{x}\mathbf{t}\,\middle|\, \begin{matrix} (\mathbf{a}_i, \overline{\alpha}_i)_{1,p} \\ (\mathbf{b}_j, \overline{\beta}_j)_{1,q} \end{matrix}\right] f(\mathbf{t})\mathbf{dt} \;\; (\mathbf{x} > 0) \tag{1}$$

and some of its modifications are studied. Here (see [1; 2; 3, ch. 1; 4]) $\mathbf{x} = (x_1, \ldots, x_n) \in \mathrm{R}^n$; $\mathbf{t} = (t_1, \ldots, t_n) \in \mathrm{R}^n$, R^n be the n-dimensional Euclidean space; $\mathbf{x} \cdot \mathbf{t} = \sum\limits_{n=1}^{n} x_n t_n$ denotes their scalar product; in particular, $\mathbf{x} \cdot \mathbf{1} = \sum\limits_{n=1}^{n} x_n$ for $\mathbf{1}= (1,\ldots,1)$. The expression $\mathbf{x} > \mathbf{t}$ means that $x_1 > t_1, \ldots, x_n > t_n$; $\int\limits_{\mathbf{0}}^{\infty} = \int\limits_{0}^{\infty} \cdots \int\limits_{0}^{\infty}$; by $\mathrm{N} = \{1, 2, \ldots\}$ we denote the set of positive integers, $\mathrm{N}_0 = \mathrm{N} \bigcup \{0\}$, $\mathrm{N}_0^n = \mathrm{N}_0 \times \cdots \times \mathrm{N}_0$; $\mathbf{k} = (k_1, \ldots, k_n) \in \mathrm{N}_0^n$ $(k_i \in \mathrm{N}_0,\ i = 1, 2, \ldots, n)$ is a multi-index with $\mathbf{k}! = k_1! \cdots k_n!$ and $|\mathbf{k}| = k_1 + \cdots + k_n$; $\mathrm{R}^n_+ = \{\mathbf{x} \in \mathrm{R}^n,\ \mathbf{x} > 0\}$; for $l = (l_1, \ldots, l_n) \in \mathrm{R}^n_+$ $\mathbf{D}^l = \frac{\partial^{|l|}}{(\partial x_1)^{l_1} \cdots (\partial x_n)^{l_n}}$; $\mathbf{dt} = dt_1 \cdots dt_n$; $\mathbf{t}^l = t^{l_1} \cdots t^{l_n}$; $f(\mathbf{t}) = f(t_1, \ldots, t_n)$. Let C^n $(n \in \mathrm{N})$ be the n-dimensional space of n complex numbers $z = (z_1, \ldots, z_n)$ $(z_j \in \mathrm{C},\ j = 1, 2, \cdots, n)$; $\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}} = \frac{d}{dx_1 \cdots dx_n}$; $\mathbf{m} = (m_1, \ldots, m_n) \in \mathrm{N}_0^n$ and $m_1 = \cdots = m_n$; $\mathbf{n} = (\overline{n}_1, \ldots, \overline{n}_n) \in \mathrm{N}_0^n$ and $\overline{n}_1 = \cdots = \overline{n}_n$; $\mathbf{p} = (p_1, \ldots, p_n) \in \mathrm{N}_0$ and $p_1 = \cdots = p_n$; $\mathbf{q} = (q_1, \ldots, q_n) \in \mathrm{N}_0$ and $q_1 = \cdots = q_n)$ $(0 \le \mathbf{m} \le \mathbf{q},\ 0 \le \mathbf{n} \le \mathbf{p})$; $\mathbf{a}_i = (a_{i_1}, \ldots, a_{i_n})$, $1 \le i \le p$, $a_{i_1}, \ldots, a_{i_n} \in \mathrm{C}$ $(1 \le i_1 \le p_1, \ldots, 1 \le i_n \le p_n)$; $\mathbf{b}_j = (b_{j_1}, \ldots, b_{j_n})$, $1 \le j \le q$, $b_{j_1}, \ldots, b_{j_n} \in \mathrm{C}$ $(1 \le j_1 \le q_1, \ldots, 1 \le j_n \le q_n)$; $\overline{\alpha}_i = (\alpha_{i_1}, \ldots, \alpha_{i_n})$, $1 \le i \le p$, $\alpha_{i_1}, \ldots, \alpha_{i_n} \in \mathrm{R}_1^+$ $(1 \le i_1 \le p_1, \ldots, 1 \le i_n \le p_n)$; $\overline{\beta}_j = (\beta_{j_1}, \ldots, \beta_{j_n})$, $1 \le j \le q$, $\beta_{j_1}, \ldots, \beta_{j_n} \in \mathrm{R}_1^+$ $(1 \le j_1 \le q_1, \ldots, 1 \le j_n \le q_n)$. The function $\mathrm{H}^{\mathbf{m},\,\mathbf{n}}_{\mathbf{p},\,\mathbf{q}}\left[\mathbf{x}\mathbf{t}\,\middle|\, \begin{smallmatrix} (\mathbf{a}_i, \overline{\alpha}_i)_{1,p} \\ (\mathbf{b}_j, \overline{\beta}_j)_{1,q} \end{smallmatrix}\right]$ in the kernel of (1) is the product of one type H-functions $H_{p,\,q}^{m,\,n}[z]$ [5, Chapters 1 and 2]:

$$\mathrm{H}_{\mathbf{p},\,\mathbf{q}}^{\mathbf{m},\,\mathbf{n}}\left[\mathbf{xt}\left|\begin{array}{c}(\mathbf{a}_i,\overline{\alpha}_i)_{1,p}\\(\mathbf{b}_j,\overline{\beta}_j)_{1,q}\end{array}\right.\right]=\prod_{k=1}^{n}H_{p_k,\,q_k}^{m_k,\,\overline{n}_k}\left[x_kt_k\left|\begin{array}{c}(a_{i_k},\alpha_{i_k})_{1,p_k}\\(b_{j_k},\beta_{j_k})_{1,q_k}\end{array}\right.\right].$$

Our paper is devoted to the study of transform (1) and some of its modifications in the weighted spaces $\mathfrak{L}_{\overline{\nu},\,\overline{r}}$- summable functions $f(\mathbf{x})= f(x_1,\ldots,x_n)$ on $\mathrm{R}_+^{\mathbf{n}}$, such that

$$\|f\|_{\overline{\nu},\overline{r}}=\{\int_{R_+^1}x_n^{\nu_n\cdot r_n-1}\{\cdots\{\int_{R_+^1}x_2^{\nu_2\cdot r_2-1}$$

$$\times[\int_{R_+^1}x_1^{\nu_1\cdot r_1-1}|f(x_1,\ldots,x_n)|^{r_1}dx_1]^{r_2/r_1}dx_2\}^{r_3/r_2}\cdots\}^{r_n/r_{n-1}}dx_n\}^{1/r_n}<\infty$$

$(\overline{r}=(r_1,\ldots,r_n)\in\mathrm{R}^n,\ 1\leq\overline{r}<\infty,\ \overline{\nu}=(\nu_1,\ldots,\nu_n)\in\mathrm{R}^n,\ \nu_1=\cdots=\nu_n)$.

Some functional and compositional properties of the integral transformation (1) and of its modifications in spaces $\mathfrak{L}_{\overline{\nu},\,\overline{2}}$ ($\overline{2}=(2,\ldots,2)$, $\overline{\nu}=(\nu_1,\ldots,\nu_n)\in\mathrm{R}^n$, $\nu_1=\cdots=\nu_n$) have been studied in the works [1, 2, 4]. We continue this research. Mapping properties such as the boundedness, the range, the representation and the inversion of the transform (1) in weighted spaces $\mathfrak{L}_{\overline{\nu},\,\overline{r}}$ are established. The results presented generalize those obtained in [5, Chapter 4.1] for one-dimensional case.

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ON A MODEL OF IMMUNE RESPONSE IN PLANTS DESCRIBED BY DELAY DIFFERENTIAL EQUATIONS

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We consider a model of immune response in plants described by the system of differential equations with two delays [1]:

$$\begin{cases} \dfrac{d}{dt}P(t) = k\Big(S(t) + W(t)\Big) \\ \qquad\qquad -ke^{-\varepsilon\tau_1}\Big(S(t-\tau_1) + W(t-\tau_1)\Big) - \varepsilon P(t), \\ \dfrac{d}{dt}S(t) = ke^{-\varepsilon\tau_1}\Big(S(t-\tau_1) + W(t-\tau_1)\Big) \\ \qquad\qquad -S(t)\Big(\lambda I(t) + \delta I(t-\tau_2) + \varepsilon S(t)\Big), \\ \dfrac{d}{dt}I(t) = I(t)\Big(\lambda S(t) - (z+\sigma) - \delta\phi I(t-\tau_2)\Big), \\ \dfrac{d}{dt}R(t) = \sigma I(t) + \delta\phi I(t) I(t-\tau_2) - \varepsilon R(t), \\ \dfrac{d}{dt}W(t) = \delta S(t) I(t-\tau_2) - \varepsilon W(t). \end{cases}$$

Here $P(t)$ is the number of proliferating cells, $S(t)$ is the number of susceptible cells, $I(t)$ is the number of infected cells, $R(t)$ is the number of recovered cells, and $W(t)$ is the number of warned cells. The delay parameter $\tau_1 \geq 0$ is responsible for the maturation time of a cell and the delay parameter $\tau_2 \geq 0$ is the time delay of the immune system's response to virus infection. The coefficients of the system are assumed to be constant and nonnegative. For more detailed description of the model, see [1].

We study the asymptotic stability of two equilibrium points corresponding to the state of the system in case of infection and the state of the system in case of recovery. We indicate estimates for the attraction sets of these equilibrium points, i.e., we find the conditions for the initial cell numbers at which the plant becomes infected and the conditions for the initial cell numbers at which the plant recovers. We establish estimates of solutions characterizing the rate of infection and the rate of recovery. When obtaining the results, various Lyapunov–Krasovskii functionals are used [2].

The results are published in [3, 4].

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BOUNDARY VALUE PROBLEMS FOR A SPECIAL CLASS OF DEGENERATE HYPERBOLIC EQUATIONS

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In the work, the solvability of boundary value problems for a special class of degenerate second-order hyperbolic equations is studied. The equations' peculiarity is that they have two independent variables, each of them can be considered as time variable. The purpose of the work is to prove existence and uniqueness of regular solutions, i.e., solutions having all Sobolev generalized derivatives included in the appropriate equation.

STOCHASTIC MODELING OF CONDITIONAL PROCESSES OF SPECIAL TYPE

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To solve problems of forecasting and interpolation of meteorological processes and fields, the method of stochastic modeling of conditional Gaussian and non-Gaussian processes and fields is often used. In the Russian literature, significant attention has been paid to these issues; there is a large number of works related to the modeling of conditional processes with point conditions. Work [2] describes algorithms for modeling conditional Gaussian processes for fixed values at given points. The methods are based on the following methods: the well-known Kriging method, a method for modeling a Gaussian vector with a conditional mean and a conditional covariance matrix using the Cholesky decomposition of the conditional covariance matrix. An earlier paper [4] presents an algorithm for modeling Gaussian conditional processes and fields that does not require the Cholesky decomposition of the conditional covariance matrix. Work [6] describes an algorithm for modeling conditional non-Gaussian processes and fields with given one-dimensional distributions and point conditions. Work [3] also discusses the issues of modeling processes with interval conditions. To solve problems of numerical stochastic modeling of Gaussian and non-Gaussian conditional processes and fields with point conditions, in the simplest case it is necessary to know mathematical expressions for conditional one-dimensional probability distributions. If they are known, then modeling of conditional processes can be carried out using the well-known method of “conditional distributions” [1, 5, 7].

This paper examines some models of multivariate distributions associated with mixtures of normal distributions, obtains expressions for the corresponding multivariate distributions, and describes algorithms for modeling certain types of conditional processes. We considered the representation of the multidimensional distribution density of a vector consisting of two subvectors in the form of a product of two densities. The first is a weighted sum of two multivariate normal densities, and the second is a weighted sum of two multivariate conditional normal densities. The number of variables in these densities corresponds to the dimension of the subvectors of the vector under consideration. The paper obtained an expression for the final density, consisting of a weighted sum of four normal distributions, and also investigated the properties of this distribution. In particular, it is shown that for a certain class of covariance matrices, the final probability distribution has the form of a superposition of two normal distributions. The paper presents algorithms for modeling conditional and unconditional random vectors that take into account the specifics of the distributions considered. These algorithms are significantly more economical than algorithms based on the elimination method [1], which is usually used to model conditional processes. The resulting distributions, in particular, can be used to approximate the distributions of hydrometeorological parameters.

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ON THE SOLVABILITY OF SOME CLASSES OF NON-LOCAL PROBLEMS FOR HIGH ORDER SOBOLEV TYPE EQUATIONS

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The report presents new results on the solvability of non-local boundary value problems for Sobolev type differential equations of the form

$$\frac{\partial^p}{\partial t^p}(Au) + Bu = f(x,t)$$

with second order operators A and B on spatial variables. A special feature of the problems is that the nonlocal condition in them is a generalized Samarsky–Ionkin condition.

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TWO-MAGNON SYSTEM IN THE FOUR-SPIN EXCHANGE HAMILTONIAN

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In crystals, it is also necessary to take into account that in addition to the two-spin exchange, there is also a multi-spin exchange. In the general case, the isotropic exchange Hamiltonian has the form

$$H' = -\sum_n \sum_{<f>} J_n(f_1, f_2, \ldots, f_n)(\overrightarrow{S}_{f_1}\overrightarrow{S}_{f_2})\ldots(\overrightarrow{S}_{f_{2n-1}}\overrightarrow{S}_{f_{2n}}).$$

For the first time, attention was drawn to the existence of multi-spin exchange in [1] when analyzing the quasi-polar model of metal, although, in fact, the considerations given there were based only on the properties of the symmetry of the exchange interaction. Here we consider two-magnon system with four-exchange Hamiltonian. Hamiltonian of the considering system has the form

$$H'' = J\sum_{m,\tau}(\overrightarrow{S}_m\overrightarrow{S}_{m+\tau})(\overrightarrow{S}_{m+2\tau}\overrightarrow{S}_{m+3\tau}),$$

where $J < 0$ is a parameter, $\overrightarrow{S}_m = (S_m^x, S_m^y, S_m^z)$ is the atom spin operator in the site $m \in Z^\nu$, $\tau = \pm e_j, j = 1, 2, \ldots, \nu$, here e_j are unit mutually orthogonal vectors. Hamiltonian H'' acts in the symmetric Fock space $\mathcal{H}_{symm}$. We let φ_o denote the vector, called the vacuum, uniquely defined be the conditions $S_m^+\varphi_0 = 0$, and $S_m^z\varphi_0 = s\varphi_0$, where $||\varphi_0|| = 1$. We set $S_m^\pm = S_m^x \pm iS_m^y$, where S_m^+ and S_m^- are the magnon creation and annihilation operators at the site m. The vectors $S_m^- S_n^-\varphi_0$ describes the state of the system of two magnons at the sites m and n with spin s. The vector space spanned be them denoted by $\mathcal{H}_2$. We denote the restriction of H'' to the space $\mathcal{H}_2$ by H_2''.

Theorem 1. *The space $\mathcal{H}_2$ is invariant under the operator H''. The operator $H_2^{''}$ is a bounded self-adjoint operator; it generates a bounded self-adjoint operator $\overline{H}_2^{''}$ acting in the space $l_2((Z^\nu)^2)$ as*

$$(\overline{H}_2^{''}f)(p,q) = J\sum_{p,q,\tau}\{[2s^2\delta_{p,q+2\tau} + 2s^2\delta_{p+2\tau,q} + s^2\delta_{p+\tau,q} + s^2\delta_{p,q+\tau}$$

$$+s^2\delta_{p+3\tau,q}+s^2\delta_{p,q+3\tau}]f(p,q)+(-s^2\delta_{p+3\tau,q}-2s^2\delta_{p+2\tau,q}-s^2\delta_{p+\tau,q})f(p-\tau,q)$$
$$+(-s^2\delta_{p,q+3\tau}-2s^2\delta_{p,q+2\tau}-s^2\delta_{p,q+\tau})f(p,q-\tau)$$
$$+(-s^2\delta_{p+3\tau,q}-2s^2\delta_{p+2\tau,q}-s^2\delta_{p+\tau,q})f(p+\tau,q)$$
$$+(-s^2\delta_{p,q+3\tau}-2s^2\delta_{p,q+2\tau}-s^2\delta_{p,q+\tau})f(p,q+\tau)+2s^2\delta_{p+2\tau,q}f(p-\tau,q-\tau)$$
$$+(s^2\delta_{p+3\tau,q}+s^2\delta_{p,q+\tau})f(p+\tau,q-\tau)+(s^2\delta_{p,q+3\tau}+s^2\delta_{p+\tau,q})f(p-\tau,q+\tau)$$
$$+2s^2\delta_{p+2\tau,q}f(p+\tau,q+\tau)\},$$

where $\delta_{k,j}$ is the Kronecker symbol. The operator $H_2^{''}$ acts on the vector $\psi \in \mathcal{H}_2$ by the formula $H_2^{''}\psi = \sum_{p,q}(\overline{H}_2^{''}f)(p,q)S_p^- S_q^- \varphi_0$.

Theorem 2. *The Fourier transform of operator $\overline{H}_2^{''}$ is an operator $\widetilde{H}_2^{''} = \mathcal{F}\overline{H}_2^{''}\mathcal{F}^{-1}$ acting in the space $L_2^{symm}((T^\nu)^2)$ be the formula*

$$(\widetilde{H}_2^{''}f)(\lambda,\mu) = J\sum_{i=1}^{\nu}\int_{T^\nu} f(s,\Lambda-s)\{8s^2\cos(\Lambda-2s)\cos(\Lambda-2\lambda)$$
$$+4s^2\cos(\frac{3\Lambda}{2}-3s)\cos(\frac{3\Lambda}{2}-3\lambda)-4s^2\cos(\frac{\Lambda}{2}-2s)\cos(\frac{3\Lambda}{2}-3\lambda)$$
$$-4s^2\cos s\cos(\Lambda-2\lambda)-4s^2\cos(2\Lambda-3s)\cos(\Lambda-2\lambda)$$
$$-4s^2\cos(\frac{3\Lambda}{2}-2s)\cos(\frac{\Lambda}{2}-\lambda)-4s^2\cos(\Lambda-3s)\cos(\Lambda-\lambda)$$
$$+4s^2\cos 2s\cos(\frac{\Lambda}{2}-\lambda)+4s^2\cos(\frac{3\Lambda}{2}-3s)\cos(\frac{\Lambda}{2}-\lambda)$$
$$-4s^2\cos(\Lambda-s)\cos(\Lambda-2\lambda)-4s^2\cos(\frac{3\Lambda}{2}-2s)\cos(\frac{3\Lambda}{2}-3\lambda)$$
$$+4s^2\cos(\Lambda-2\lambda-s)\cos(\Lambda-2\lambda)+4s^2\cos(2\Lambda-2s)\cos(\Lambda-2\lambda)\}ds.$$

Theorem 3. *Let $\nu = 1$. The continuous spectrum of the operator $\widetilde{H}_2^{''}$ is consists of the point 0 and discrete spectrum of the operator $\widetilde{H}_2^{''}$ is consists of no more than six eigenvalues.*

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VISCOSITY SOLUTIONS OF EQUATIONS WITH NON-STANDARD GROWTH CONDITIONS

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In the report we will consider the following equation with non-standard growth conditions, which has a large number of applications in mechanics:

$$u_t - \sum_{i=1}^{n} (|u_{x_i}|^{p_i(t)-2} u_{x_i})_{x_i} = \mathcal{B}(t, x, u, \nabla u) \quad \text{in} \quad \Omega_T = (0,T) \times \Omega. \qquad (1)$$

For now there is an extensive literature concerning the existence, uniqueness and qualitative behavior of solutions to boundary value problems for (1). When studying equations of the form (1), methods of the calculus of variations, various topological, and also approximation methods are used. Due to the degeneracy and even possible singularity of equations of the form (1), solutions are sought in the class of weak solutions, mainly Sobolev solutions. In this regard, we note the monograph by S. Antontsev and S. Shmarev [1].

As noted above, one of the ways to study the existence of solutions to boundary value problems for (1) is an approximation one, which we, in particular, use to prove the existence of solutions to boundary value problems for (1) that have high smoothness. To this end, we regularize the original problem and prove the existence of classical solutions to regularized problems. Using the Minty–Browder monotonicity method, it is possible to carry out the passage to the limit in the family of classical solutions of regularized problems and obtain solvability in the class of Sobolev solutions. But this can only be done if $\mathcal{B}(t, x, u, \nabla u)$ is linear in the gradient. Otherwise, the obtained a priori estimates of classical solutions of regularized problems do not make it possible to make such a passage to the limit.

This problem can essentially be overcome using the theory of viscosity solutions, for which the passage to the limit can be carried out with much weaker a priori estimates. Viscosity solutions belong to the category of weak solutions, but are defined not in an integral sense, like Sobolev solutions, but pointwise, through smooth sub- and supersolutions of the original equation. These smooth sub- and supersolutions, as in the theory of weak Sobolev solutions, play the role of test functions.

Using this tools, it is possible to prove existence theorems even for $\mathcal{B}(t, x, u, \nabla u)$, which does not satisfy the Bernstein–Nagumo condition, that is, having an arbitrary growth with respect to the gradient. The use of the tools of the theory of viscosity solutions allowed us to avoid the procedure of passing to the limit in the gradient terms. The existence theorem is proven based on the following fact: the uniform limit of a sequence of viscosity solutions is also a viscosity solution.

Thus, we were able to prove the existence and uniqueness of a Lipschitz-continuous viscosity solution to the first initial-boundary value problem for equations of the form (1) in the case where $\mathcal{B}(t, x, u, \nabla u)$ has arbitrary growth in gradient [2].

The obtained results are closely related to the theory of the existence of classical solutions of elliptic and parabolic equations.

The study was carried out within the framework of the state contract of the Sobolev Institute of Mathematics (project no. FWNF-2022-0008).

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ULTRAPARABOLIC EQUATIONS WITH DEGENERACY

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Let Ω and G be bounded domains in spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ of variables $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ respectively, $c(x, y)$, $f(x, y)$, $\alpha^k(y)$, $k = 1, \ldots, m$, are given functions defined when $x \in \overline{\Omega}$, $y \in \overline{G}$, Δ is a Laplace operator with respect to variables $x_1, \ldots, x_n$. We study the solvability of various boundary value problems for differential equations

$$\Delta u + \sum_{k=1}^{m} \alpha^k(y) u_{y_k} = f(x, y), \quad (x, y) \in Q.$$

We prove theorems of the existence and uniqueness of regular solutions, i.e., solutions having all Sobolev generalized derivatives included in the equation. Some properties of solutions are also studied.

NONLOCAL BOUNDARY VALUE PROBLEMS FOR ULTRAHYPERBOLIC EQUATIONS

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We study of the solvability of nonlocal boundary value problems for differential equations

$$u_{xt} - au_{xx} - \Delta_y u + c(x, y, t)u = f(x, y, t)$$

($0 < x < 1$, $0 < t < T < +\infty$, $y \in \Omega \subset \mathbb{R}^m$, Δ is the Laplace operator in the space of variables $y_1, \dots, y_m$, a is a real number). The work aim is to prove the existence and uniqueness of regular solutions to the problems under study, i.e., solutions that have all Sobolev derivatives included in the equation.

SPHERICAL POLYHARMONIC EQUATION AND WEIGHTED CUBATURE FORMULAS

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Let S be the sphere of unit radius in R^n, $n \geq 2$. The projection of an arbitrary point x in R^n, $x \neq 0$, to S will be denoted by θ; i.e., we assume that $\theta = x/\rho$, where $\rho = |x|$. So θ is a point in S. In what follows, the integrals over θ are surface integrals over dS. Let us consider the differential equation of the form

$$(-D)^m u(\theta) = p(\theta), \tag{1}$$

where D is the Laplace–Beltrami operator with respect to $d\theta$ [1], m is a positive integer, and $p(\theta)$ is a continuous function on S which obey the orthogonality condition $\int p(\theta)\, d\theta = 0$. The main results of the talk are about the solutions to (1). They are formulated in the following two theorems [2].

Theorem 1. *Let m be an integer and $m > (n-1)/2$. Then for every functional $l(\theta)$ in $C^*(S)$ with $(l, 1) = 0$, the problem*

$$(-D)^m u(\theta) = l(\theta), \qquad \int u(\theta)\, d\theta = 0, \tag{2}$$

has a unique solution $u(\theta)$ in the spherical Sobolev space H^m. For $m \geq (3n-2)/4$ the solution to (2) belongs to the space $C^{(2m-3n/2+1)}(S)$.

The expansion of $u(\theta)$ in the series has the form

$$u(\theta) = \sum_{k=1}^{\infty} \frac{1}{k^m (n+k-2)^m} \sum_{l=1}^{\sigma(k)} (l, Y_{k,l}) Y_{k,l}(\theta).$$

Here the set of functions $\{Y_{k,\,l}(\theta) \mid l = 1, 2, \ldots, \sigma(k)\}$ constitute an orthonormal basis for the space of spherical harmonics of order k:

$$\int Y_{k,l}(\theta) Y_{k,p}(\theta)\, d\theta = \delta_l^p.$$

Theorem 2. *Let $p(\theta)$ be a member of the spherical Sobolev space H^s for some $s > (n-1)/2$ and the equality $\int p(\theta)\,d\theta = 0$ holds. Then there is a unique solution to the spherical polyharmonic equation*

$$(-D)^m u(\theta) = p(\theta)$$

such that it is orthogonal to the identically-one function and belongs to the space H^q for $q = s + 2m$. The function $u(\theta)$ can be written as follows

$$u(\theta) = \int G(\theta \cdot \theta')p(\theta')\,d\theta',$$

where the function $G(\theta \cdot \theta')$ is the Green's function of $(-D)^m$.

The definition of $G(\theta \cdot \theta')$ is as follows

$$G(\theta \cdot \theta') = \frac{1}{\sigma_{n-1}} \sum_{k=1}^{\infty} \frac{\sigma(k)}{k^m(n+k-2)^m} G_k^{(n)}(\theta \cdot \theta'). \tag{3}$$

Here $G_k^{(n)}$ is the normalized Hegenbauer polynomial.

For $s > (n-1)/2$ the series on the right-hand side of (3) converges absolutely and uniformly. For two points θ and $\theta^{(j)}$ in S the function $G(\theta \cdot \theta^{(j)})$ is a solution to the equation

$$(-D)^m G(\theta \cdot \theta^{(j)}) = \delta(\theta - \theta^{(j)}) - \frac{1}{\sigma_{n-1}} \int \delta(\theta - \theta')\,d\theta'.$$

Spherical polyharmonic equation (1) with error functionals in the right hand side is very important in the theory of cubature formulas [3–4].

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SOLVABILITY OF LYAPUNOV-TYPE MATRIX EQUATIONS

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We study the relationship between the solvability of matrix equations of the form

$$\sum_{j,k=0}^{N} \alpha_{jk}(A^*)^j H A^k = C, \tag{1}$$

where A is $(n\times n)$-matrix, the right side C is $(n\times n)$-matrix too, α_{jk} are numerical coefficients, and belonging of the spectrum of matrix A to sets lying inside or outside domains bounded by an ellipse or a parabola. Conditions for perturbations of matrix elements are obtained, which guarantee that the spectrum of matrix $A+B$ belongs to the mentioned domains. Under these conditions on perturbations B, corresponding matrix equations

$$\sum_{j,k=0}^{N} \alpha_{jk}((A+B)^*)^j H (A+B)^k = C$$

are uniquely solvable.

Some theorems on the solvability of matrix equations of the form (1) are contained in [1–4].

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OUTPUT FEEDBACK STABILIZATION OF LINEAR SYSTEMS WITH MULTIPLE DELAYS USING MODEL REDUCTION METHODS

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This paper investigates the output feedback for large-scale linear systems with multiple delays. Firstly, we obtain a structure-preserving low-dimensional time-delay system by using the two-level orthogonal Arnoldi process. Then, we transform the state of the low dimensional time-delay system into the output variables of the system through linear transformation, thus the input-output relationship of the system is established directly, and the output feedback controller is designed for this system. Finally, based on the argument principle, we examine the stability of the closed-loop system. And a numerical example is exhibited to verify the efficiency of the algorithms.

Consider the following linear system with multiple delays:

$$\begin{cases} \dot{x}(t) = A_0 x(t) + \sum_{i=1}^{q} A_i x(t-\tau_i) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{1}$$

where $A_0, A_i \in \mathbb{R}^{n\times n}, i = 1, 2, \ldots, q$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$. $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, and $y(t) \in \mathbb{R}^p$ is the output vector. $\tau_i, i = 1, 2, \ldots, q$, are time-delays, and we assume that $0 < \tau_1 < \tau_2 < \ldots < \tau_q$.

For a large-scale unstable time-delay system (1), we consider an output feedback controller with the following structure:

$$u(t) = -F_0 y(t) - \sum_{i=1}^{q} F_i y(t-\tau_i), \tag{2}$$

where $F_0, F_i \in \mathbb{R}^{m\times p}$ $(i = 1, 2, \ldots, q)$ are output feedback gain matrices. The number of output variables of the system is generally less than that of the state variables, and all of them can be measured directly. Therefore, the output feedback control is easy to realize in engineering practice. From (1) and (2), we obtain the closed-loop system

$$\dot{x}(t) = (A_0 - BF_0C)x(t) + \sum_{i=1}^{q}(A_i - BF_iC)x(t - \tau_i). \tag{3}$$

We now present the main result in this paper.

Algorithm.

Step 1. Applying the two-level orthogonal Arnoldi process to the system (1), we can obtain the standard orthogonal basis of Krylov subspace.

Step 2. Using the subspace projection technique, we directly project the state of system (1) onto the subspace to obtain a reduced time-delay system, and the state of reduced time-delay system is transformed into the output variables of system through linear transformation.

Step 3. We apply optimization techniques to design the output feedback controller to stabilize the low dimensional time-delay system.

Step 4. Substituting the output feedback controller into system (1), we utilize the argument principle (see [3], Algorithm 1) to check the stability of the closed-loop system (3). If the closed-loop system is asymptotically stable, we complete the controller design.

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DELAY-DEPENDENT STABILITY OF RUNGE–KUTTA METHODS WITH TIME-ACCURATE AND HIGHLY-STABLE EXPLICIT OPERATORS FOR NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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In this paper, we studied the delay-dependent stability of neutral delay differential equations (NDDEs). Runge–Kutta methods with time-accurate and highly-stable explicit operators (TASE-RK) are proposed to solve the NDDEs. By applying the argument principle, we obtained sufficient conditions for delay-dependent stability of TASE-RK.

The NDDEs are described as

$$u'(t) = Lu(t) + \sum_{j=1}^{m}[M_j u(t-\tau_j) + N_j u'(t-\tau_j)], \quad \sum_{j=1}^{m} ||N_j|| < 1,$$

where $u(t) \in \mathbb{R}^d$, $L, M_j, N_j \in \mathbb{R}^{d\times d}$ are parameter matrices, $\tau_j > 0$ ($j = 1, \dots, m$), and $\tau_m > \tau_{m-1} > \dots > \tau_1$.

For a linear model $\frac{dY}{dt} = LY$, reference [2] proposed a family of operators $T_L^{(p)}$ such that $\frac{dY}{dt} = (T_L^{(p)})Y$ is nearly unconditionally stable by using s-stage p-th order explicit RK,

$$T_L^{(p)}(\alpha, h) = \begin{cases} (1-\alpha hL)^{-1}, & p = 1, \\ \dfrac{2^{p-1}T_L^{(p-1)}(\frac{\alpha}{2}, h) - T_L^{(P)}(\alpha, h)}{2^{p-1}-1}, & p \geq 2, \end{cases}$$

where h is time-step, $\alpha > 0$ is parameter which guarantees the stability of the explicit RK methods. $T_L^{(p)} = 1 + O(h^p)$ ensures the time accuracy.

We extend this method to NDDEs,

$$u_{n+1} = u_n + \sum_{i=1}^{s} b_i K_{n,i},$$

$$K_{n,i} = T^{(p)}_{a+b+c} \cdot (hL(u_n + \sum_{j=1}^{i-1} a_{ij} K_{n,j}) + h \sum_{k=1}^{m} M_k (u_{n-m} + \sum_{j=1}^{i-1} a_{ij} K_{n-m,j})$$
$$+ \sum_{k=1}^{m} N_k K_{n-m,i}), \quad i = 1, 2, \ldots, s.$$

It has stronger stability than explicit RK when $\alpha > 0$. Furthermore, the TASE1-RK1 is unconditionally stable when $\alpha > 0.5$. Its characteristic polynomial is

$$P(z) = \det \left\{ \begin{bmatrix} I_{sd} - h(A \bigotimes T^{(p)}_{a+b+c} L) & 0 \\ -b^{\mathrm{T}} \bigotimes I_d & I_d \end{bmatrix} z^{m+1} \right.$$
$$- \begin{bmatrix} 0 & h(e \bigotimes T^{(p)}_{a+b+c} L) \\ 0 & I_d \end{bmatrix} z^m$$
$$\left. - \sum_{k=1}^{m} \begin{bmatrix} (h(A \bigotimes T^{(p)}_{a+b+c} M_k) + I_s \bigotimes T^{(p)}_{a+b+c} N_k) z & h(e \bigotimes T^{(p)}_{a+b+c} M_k) \\ 0 & 0 \end{bmatrix} \right\}.$$

Theorem. *For a TASE-RK method, assume that*
(1) the NDDEs are asymptotically stable;
(2) the TASE-RK method is natural;
(3) the characteristic polynomial $P(z) \neq 0$ on $\mu = \{z : |z| = 1\}$ and

$$\frac{1}{2\pi} \triangle_\mu \arg P(z) = d(s+1)(m+1).$$

then the TASE-RK method for NDDEs is delay-dependent stability.

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ESTIMATES OF SOLUTIONS TO ONE INHOMOGENEOUS SYSTEM OF FUNCTIONAL DIFFERENCE EQUATIONS

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We consider a system of linear inhomogeneous functional difference equations with constant coefficients and two delays

$$y(t) = A_1 y(t-\tau_1) + A_2 y(t-\tau_2) + f(t), \quad t > 0, \tag{1}$$

where A_j, $j = 1, 2$, are constant $(n \times n)$-matrices, $\tau_1 = l\tau_2$, $l > 0$ is integer, $\tau_2 > 0$, $f(t)$ is a continuous vector function satisfying the estimate

$$\|f(t)\| \leq ae^{-bt}, \quad a, b > 0.$$

Without using information about the spectrum of matrices A_1 and A_2, we establish estimates for solutions $y(t)$ to system (1) for $t > 0$. We indicate conditions on matrices A_1 and A_2, under which the solutions to system (1) tend to zero as $t \to +\infty$ at an exponential rate.

ON PROPERTIES OF SOLUTIONS TO ONE INHOMOGENEOUS SYSTEM OF FUNCTIONAL DIFFERENCE EQUATIONS

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We consider a system of inhomogeneous functional difference equations

$$x(t) = Ax(t-\tau) + f(t), \quad t > 0, \tag{1}$$

where A is constant $(n \times n)$-matrix, all eigenvalues of which belong to the unit circle, $f(t)$ is a continuous vector function satisfying the estimate

$$\|f(t)\| \leq \alpha e^{-\gamma t}, \quad \alpha, \gamma > 0.$$

Our aim is to obtain estimates for solutions $x(t)$ to system (1) for $t > 0$ and to investigate the influence of perturbations on the behavior of the solutions.

Using a solution to the discrete Lyapunov equation [1], estimates characterizing the exponential decay of solutions to system (1) at infinity are obtained. Systems with perturbations are considered and estimates of solutions for perturbed systems are established. Statements about continuous dependence of solutions to the initial value problems for the considered systems of inhomogeneous functional difference equations are proved.

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ESTIMATES OF SOLUTIONS TO DELAY DIFFERENTIAL EQUATIONS

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We consider a class of systems of differential equations with distributed delay

$$\frac{d}{dt}y(t) = A(t)y(t) + \int\limits_{t-\tau_2}^{t-\tau_1} B(t, t-s)y(s)\,ds, \tag{1}$$

where $A(t)$ is a matrix of dimension $n \times n$ with continuous T-periodic elements, $B(t,s)$ is a matrix of dimension $n \times n$ with continuous T-periodic with respect to first variable elements, $\tau_2 > \tau_1 > 0$ are constants.

Sufficient conditions for the exponential stability of the zero solution to (1) are given, estimates of solutions that characterize the exponential decrease at infinity are established. We use a functional

$$v(t,y) = \langle H(t)y(t), y(t)\rangle + \int\limits_{\tau_1}^{\tau_2}\int\limits_{t-\eta}^{t} \langle K(t-s,\eta)y(s), y(s)\rangle\, ds d\eta$$

that is an analogue to the functionals introduced in [1, 2].

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TWO PHASE-FIELD MODELS FOR SOLID-SOLID PHASE TRANSFORMATIONS DRIVEN BY CONFIGURATIONAL FORCES

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This talk is based on some recent work by the speaker of this talk and his colleagues. We present two new phase-field models, which were formulated in [1, 2] by H.-D. Alber and the speaker, for phase transitions driven by configurational forces in elastically deformable solids. These models consist of a linear elasticity subsystem coupled to a nonlinear, degenerate parabolic equation of second or fourth order, and the two models differ, respectively, from the well-known Allen–Cahn and Cahn–Hilliard models by a non-smooth gradient term of an order parameter. Some numerical and theoretical results about one of these models are then stated, and we refer the reader to [3–6] for more results. Their applications include: (1) To describe martensitic phase transitions in, e.g., shape memory alloys, and (2) To model sintering in powder metallurgy.

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DIFFERENTIAL AND DIFFERENCE EQUATIONS

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ABSTRACTS